Kinematic trajectories for vehicle agents

Rasmus Fonseca & Tim Caldwell

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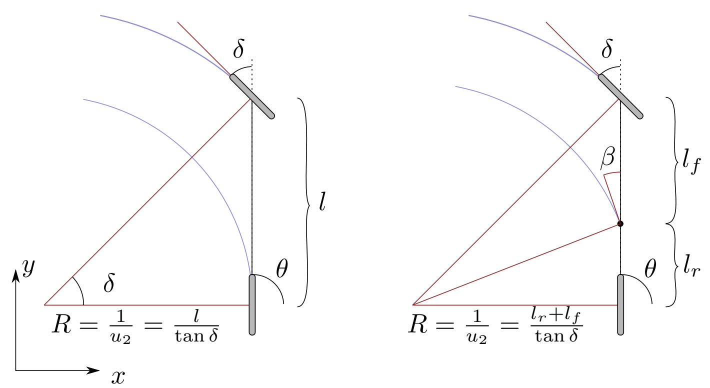
Abstract

Kinematic trajectories are used to move agents forward along a predicted route frame. Here we describe two underlying kinematic models based on bicycle models and how they're translated from cartesian into route coordinates.

1 Kinematic bicycle models

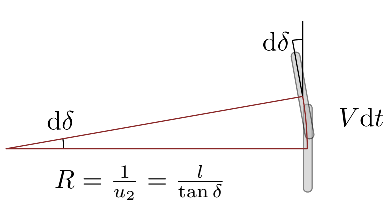
The commonly used simplification of a car is the rear-axle centered bicycle model:



Our vehicle model uses lateral accelleration (u_1) and curvature of the front-wheel turning (u_2) as controls. The geometry of the rear-axle centered model (left) can be described as

$$\dot{V} = u_1 \tag{1}$$
$$\dot{x} = V\cos\theta \tag{2}$$
$$\dot{y} = V\sin\theta \tag{3}$$
$$\dot{\theta} = V\frac{\tan\delta}{l} \tag{4}$$
$$= Vu_2 \tag{5}$$

Where (x, y) are the coordinates of the rear-axle and V the velocity. Derivatives of x, y, and V are straightforward, and $\dot{\theta}$ can be found by observing that the rear-axle will travel by the distance $V\mathrm{d}t$ along the circle which correponds to an angular change of $\frac{V\mathrm{d}t}{R} = Vu_2\mathrm{d}t$.



Polack et al. [1] outline a similar model which is instead centered somewhere on the middle of the vehicles axis (right side of first figure). Here, the cartesian geometry can be described using the slip-angle of the center-of-mass:

$$\beta(\delta) = \mathrm{atan}\left(\tan(\delta)\frac{l_r}{l_r+l_f}\right) \tag{6}$$
$$\dot{V} = u_1 \tag{7}$$
$$\dot{x} = V\cos(\theta+\beta(u_2)) \tag{8}$$
$$\dot{y} = V\sin(\theta+\beta(u_2)) \tag{9}$$
$$\dot{\theta} = V\frac{\sin\beta(u_2)}{l_r} \tag{10}$$

By substituting $u_2 = \frac{\tan\delta}{l_r+l_f}$ into β and simplifying using trigonometric equations [2, 3] all calls to atan can be eliminated:

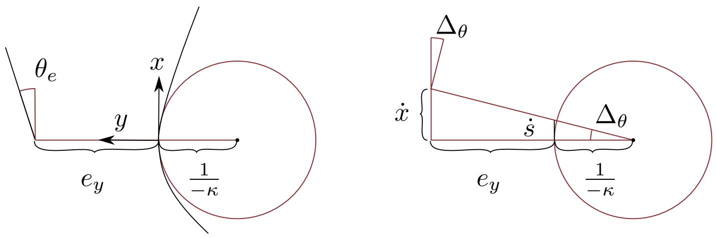
$$\beta(u_2) = \mathrm{atan}\left(u_2 l_r\right) \tag{11}$$
$$\dot{\theta} = V\frac{\sin\beta(u_2)}{l_r} \tag{12}$$
$$= V\frac{\sin\mathrm{atan}(u_2 l_r)}{l_r} \tag{13}$$
$$= V\frac{u_2}{\sqrt{1+u_2^2 l_r^2}} \tag{14}$$
$$= V\psi u_2 \tag{15}$$
$$\dot{x} = V\cos(\theta+\beta(u_2)) \tag{16}$$
$$= V\left(\cos(\theta)\cos(\mathrm{atan}(u_2 l_r)) - \sin(\theta)\sin(\mathrm{atan}(u_2 l_r))\right) \tag{17}$$
$$= V\left(\frac{\cos(\theta)}{\sqrt{1+u_2^2 l_r^2}} - \frac{\sin(\theta)u_2 l_r}{\sqrt{1+u_2^2 l_r^2}}\right) \tag{18}$$
$$= V\frac{(\cos(\theta)-\sin(\theta)u_2 l_r)}{\sqrt{1+u_2^2 l_r^2}} \tag{19}$$
$$= V\psi\left(\cos(\theta)-\sin(\theta)u_2 l_r\right) \tag{20}$$
$$\dot{y} = V\sin(\theta+\beta(u_2)) \tag{21}$$
$$= V\left(\sin(\theta)\cos(\mathrm{atan}(u_2 l_r)) + \cos(\theta)\sin(\mathrm{atan}(u_2 l_r))\right) \tag{22}$$
$$= V\left(\frac{\sin(\theta)}{\sqrt{1+u_2^2 l_r^2}} + \frac{\cos(\theta)u_2 l_r}{\sqrt{1+u_2^2 l_r^2}}\right) \tag{23}$$
$$= V\frac{(\sin(\theta)+\cos(\theta)u_2 l_r)}{\sqrt{1+u_2^2 l_r^2}} \tag{24}$$
$$= V\psi\left(\sin(\theta)+\cos(\theta)u_2 l_r\right) \tag{25}$$

where

$$\psi = \frac{1}{\sqrt{1+u_2^2 l_r^2}} \tag{26}$$

2 Transformation to road-reference coordinates

Route reference lines are composed of geometric primitives that can be slices of either piecewise clothoid curves, circles, or lines. Positions in the route frame are represented using $(s,\, e_y)$-coordinates corresponding to distance along the curve and lateral offset. Vehicle orientation is specified using the yaw θ_e with respect to the tangent of the reference line at the s coordinate. A model of the rear-centered bicycle in



this coordinate system can be defined by extending on the cartesian set of equations where the x-axis is aligned with the tangent of the curve (left side of figure). This ensures that $\dot{s}$ is proportional to $\dot{x}$, $\dot{e_y}$ to $\dot{y}$, and that $\theta = \theta_e$. An assumption is made that for small steps the curve is well-approximated by a circle with the same curvature, κ, as the curve.

$$\dot{V} = u_1 \tag{27}$$

$$\dot{s} = \frac{\dot{x}}{1 - e_y\kappa} \tag{28}$$

$$= V\omega\cos\theta_e \tag{29}$$

$$\dot{e_y} = \dot{y} \tag{30}$$

$$= V\sin\theta_e \tag{31}$$

$$\dot{\theta_e} = \dot{\theta} - \Delta_\theta \tag{32}$$

$$= \dot{\theta} - \frac{\dot{x}}{e_y - 1/\kappa} \tag{33}$$

$$= \dot{\theta} - \kappa\omega\dot{x} \tag{34}$$

$$= V\left(u_2 - \kappa\omega\cos e_y\right) \tag{35}$$

where

$$\omega = \frac{1}{1 - e_y\kappa} \tag{36}$$

Equation 28 comes from interpreting the triangle spanned by $\dot{x}$ from the rear-axle to the center of the reference lines curvature center (right side of above figure). The relationship between $\dot{x}$ and the full baseline roughly[1] equals the relationship between $\dot{s}$ and the radius of the curvature-circle, $-1/\kappa$. Rewriting we get:

$$\frac{\dot{s}}{-1/\kappa} = \frac{\dot{x}}{e_y - 1/\kappa} \Leftrightarrow \tag{37}$$

$$\dot{s}\kappa = \frac{\dot{x}}{1/\kappa - e_y} \Leftrightarrow \tag{38}$$

$$\dot{s} = \frac{\dot{x}}{1 - e_y\kappa} \tag{39}$$

[1] More accurately, $\dot{s}$ would be the extent along the circle arc, but this introduces an expensive atan computation which we'd like to avoid.

The Δ_θ term in Equation 33 is the compensation to θ_e required to keep the same heading when stepping the distance $\dot{s}$ along the route frame. Like in the derivation of $\dot{s}$, this term is determined by observing the curvature circle, but we now assume that at small values, $\dot{s}$ divided by the full baseline roughly[2] equals the angular change around the curvature center.

The transformation from rear-axle centered vehicle to CoM can be performed by substituting Equations 15, 20, and 25 into 28, 30, and 34.

$$\dot{V} = u_1 \tag{40}$$

$$\dot{s} = V\psi\omega\left(\cos(\theta_e) - \sin(\theta_e)u_2 l_r\right) \tag{41}$$

$$\dot{e_y} = V\psi\left(\sin(\theta_e) + \cos(\theta_e)u_2 l_r\right) \tag{42}$$

$$\dot{\theta_e} = V\psi\left(u_2 - \kappa\omega\left(\cos(\theta_e) - \sin(\theta_e)u_2 l_r\right)\right) \tag{43}$$

where, to summarize

$$\psi = \frac{1}{\sqrt{1 + u_2^2 l_r^2}} \tag{44}$$

$$\omega = \frac{1}{1 - e_y\kappa} \tag{45}$$

References

[1] P. Polack, F. Altche, B. D'Andrea-Novel, and A. De La Fortelle *The Kinematic Bicycle Model: a Consistent Model for Planning Feasible Trajectories for Autonomous Vehicles.* HAL archives-ouvertes, 2017. `https://hal-polytechnique.archives-ouvertes.fr/hal-01520869`

[2] Wikipedia *Inverse trigonometric functions.* `https://en.wikipedia.org/wiki/Inverse_trigonometric_functions`

[3] Wikipedia *Angle sum and difference identities.* `https://en.wikipedia.org/wiki/List_of_trigonometric_identities#Angle_sum_and_difference_identities`

[2] Again, this is only true in the limit of small steps, but avoids an expensive atan computation.