1 About this Document

1.1 Author

These notes were originally written by Joona Kiiski, but may of course be updated by others.

1.2 Purpose

To give a quick introduction to clothoids and help to clarify the implementation details

1.3 Maintanence

This document is maintained using Lyx-editor. If you need to update the document, do the following:

- apt-get install lyx ghostscript
- Go through the basic tutorial: http://wiki.lyx.org/LyX/Tutorials
- Update the document: lyx clothoid_implementation_notes.lyx
- Re-export PDF: File -> Export -> PDF (ps2pdf)

2 Normalized Euler Spiral

2.1 Basics

Read the following:

- https://en.wikipedia.org/wiki/Euler_spiral
- https://en.wikipedia.org/wiki/Fresnel_integral

2.2 Notes

For normalized Euler Spiral (L is distance from origin):

$x = S(L) = \int_0^L \cos(s^2) \cdot ds$

$y = C(L) = \int_0^L \sin(s^2) \cdot ds$

$S(L)$ and $C(L)$ are called Fresnel integrals which need to be calculated numerically. Currently we use cephes math-library to do this. However some authors (including the author of cephes-library) prefer to represent the Fresnel integrals as:

$C(t) = \int_0^t \cos(\frac{\pi}{2}t^2) \cdot dt$

$S(t) = \int_0^t \sin(\frac{\pi}{2}t^2) \cdot dt$

However it's easy to switch between the two forms just by changing the integration variable: $\tau = \sqrt{\frac{\pi}{2}} \cdot t$

3 Arbitrary Clothoid

3.1 Definition

By an arbitrary clothoid, we mean a clothoid which can have any length, start curvature, end curvature, starting point and orientation.

3.2 Connection to Normalized Euler Spiral

An arbitrary clothoid can be constructed as follows:

- Pick a section $[s_0, s_1]$ from Euler Spiral ($s_1 > s_0$)
- Move it (translation)
- Scale it
- Rotate it
- Mirror it along x-axis (or not!)

All the above operations can be done by using elementary geometric transformations.

The inverse is also true. Any clothoid can can be converted into a $[s_0, s_1]$ section on Euler Spiral by elementary geometric transformations.

3.3 Normalization and Denormalization

When we want to solve any clothoid specific problem:

- We first perform a coordinate transformation where clothoid becomes a section on Euler spiral. We call this process "Normalization".
- Then we solve the problem - usually this involves use of Fresnel integral.
- After solving the problem we perform the inverse transformation for the result. We call this process "Denormalization".

Obviously the whole point of doing this is to be able to use Fresnel integrals for any problem.

"ClothoidSection" class in the current implementation provides functions for Normalization, Denormalization and for basic calculations in the normalized frame.

3.4 Degeneration

3.4.1 Circle Arc

If clothoid start curvature is equal to clothoid end curvature, then we cannot use normalization/denormalization technique outlined above as the scaling coefficient becomes zero/infinity. Instead we have a circle arc and we need to solve the problem using the geometric properties of a circle.

3.4.2 Line Segment

If clothoid start curvature and end curvature are both zero, then we have a straight line, and need to solve the problem using the geometric properties of a line segment.

4 Clothoid Fitting

4.1 Problem Statement

Given a sequence of (x, y) points, fit a piece-wise clothoid which is G_1 continuous and "close to" G_2 continuous. For practical reasons, (x, y) points may also contain additional tangent and curvature constraints. The resulting piece-wise clothoid needs to be "sensible", but it's impossible to give an exact mathematical definition for this.

4.2 Recipe

Our current strategy is as follows:

1. Our implementation of PCC-library provides good-looking curves which are good approximations of real clothoids. Unfortunately it doesn't provide analytical formulas. After fitting we read a tangent for each point, so that we have a sequence of triplets (x, y, ϑ), where ϑ is the tangent angle of the curve at (x, y).

2. For each consecutive triplet pair (x_n, y_n, ϑ_n) $(x_{n+1}, y_{n+1}, \vartheta_{n+1})$, fit a real clothoid between the points. Use curve provided by PCC-library as an initial guess. This provides G_1 continuous curves, but discontinuities in curvature seem to be relatively minor.

5 PCC

PCC-library has it's own documentation, so it's not in the scope of this document.

6 Single Clothoid Fit

6.1 Paper

We use the algorithm outlined in paper "Fast And Accurate Clothoid Fitting" by Enrico Bertolazzi and Marco Frego. However there are some significant and non-obvious differences in the implementation which are documented in this section.

6.2 Class

The implementation is contained in a class named "SingleClothoidFitter". All variables in the class are named to match the variables used in the paper.

6.3 Algorithm

We use algorithm that is called "FindA" in the paper. The corresponding implementation method is "SingleClothoidFitter::fit()". The key challenge of the algorithm is around calculating $g(A)$ and $g'(A)$ which are discussed next.

6.4 g(A) and h(A)

The implementation method is "SingleClothoidFitter::ThetaFunction".

In the paper $g(A)$ and $h(a)$ are defined as follows:

$g(A) = \int_0^1 \sin(A\tau^2 + (\Delta\vartheta - A)\tau + \Delta\varphi) \cdot d\tau$

$h(A) = \int_0^1 \cos(A\tau^2 + (\Delta\vartheta - A)\tau + \Delta\varphi) \cdot d\tau$

The paper goes very deep into numerical methods about solving these integrals. Instead we just note that these match exactly the formula of generalized clothoid:

$x(s) = \int_0^s \cos(\frac{1}{2}\kappa'\tau^2 + \kappa\tau + \vartheta_0) \cdot d\tau$

$y(s) = \int_0^s \sin(\frac{1}{2}\kappa'\tau^2 + \kappa\tau + \vartheta_0) \cdot d\tau$

where $\kappa'\tau + \kappa$ is curvature and ϑ_0 is the initial angle (clothoid orientation). So calculating $g(a)$ and $h(a)$ is identical to calculating $\Delta x = x_{end} - x_{start}$ and $\Delta y = y_{end} - y_{start}$ for the corresponding clothoid. Our Spiral/ClothoidSection class provides utilities to do just this, so we use them here. Of course the degenerate cases (circle, line) require special handling.

6.5 g'(A)

The implementation method is "SingleClothoidFitter::ThetaFunctionDerivative".

The most challenging thing required by the paper is the calculation of $g'(a)$. Because we've already calculated $h(a)$ and $g(a)$ numerically we want to reuse them. The formula for $g'(a)$ as a function of $h(a)$ and $g(a)$ is derived below:

Definitions:

$C(A) := h(A) = \int_0^1 \cos(A\tau^2 + (\Delta\vartheta - A)\tau + \Delta\varphi) \cdot d\tau$

$S(A) := g(A) = \int_0^1 \sin(A\tau^2 + (\Delta\vartheta - A)\tau + \Delta\varphi) \cdot d\tau$

Solve Derivative $S^{'}(A)$:

$S^{'}(A) = \int_0^1 \cos(A\tau^2 + (\Delta\vartheta - A)\tau + \Delta\varphi) \cdot (\tau^2 - \tau) \cdot d\tau$

$S^{'}(A) = \int_0^1 \cos(A\tau^2 + (\Delta\vartheta - A)\tau + \Delta\varphi) \cdot \tau^2 \cdot d\tau - \int_0^1 \cos(A\tau^2 + (\Delta\vartheta - A)\tau + \Delta\varphi) \cdot \tau \cdot d\tau$

$S'(A) = S_1^{'}(A) - S_2^{'}(A)$

6.5.1 General Case, $A \neq 0$

First $S_2'(A)$:

$S_2'(A) = \int_0^1 \cos(A\tau^2 + (\Delta\vartheta - A)\tau + \Delta\varphi) \cdot \tau \cdot d\tau$

$S_2'(A) = \frac{1}{2A} \int_0^1 2A\tau \cdot \cos(A\tau^2 + (\Delta\vartheta - A)\tau + \Delta\varphi) \cdot d\tau$

$S_2'(A) = \frac{1}{2A} \int_0^1 (2A\tau + \Delta\vartheta - A - \Delta\vartheta + A) \cdot \cos(A\tau^2 + (\Delta\vartheta - A)\tau + \Delta\varphi) \cdot d\tau$

$S_2'(A) = \frac{1}{2A} \int_0^1 [(2A\tau + \Delta\vartheta - A) \cdot \cos(A\tau^2 + (\Delta\vartheta - A)\tau + \Delta\varphi) + (-\Delta\vartheta + A) \cdot \cos(A\tau^2 + (\Delta\vartheta - A)\tau + \Delta\varphi)] \cdot d\tau$

$S_2'(A) = \frac{1}{2A} \int_0^1 (2A\tau + \Delta\vartheta - A) \cdot \cos(A\tau^2 + (\Delta\vartheta - A)\tau + \Delta\varphi) \cdot d\tau + \frac{1}{2A} \int_0^1 (-\Delta\vartheta + A) \cdot \cos(A\tau^2 + (\Delta\vartheta - A)\tau + \Delta\varphi) \cdot d\tau$

$S_2'(A) = \frac{1}{2A} \Big|_0^1 \sin(A\tau^2 + (\Delta\vartheta - A)\tau + \Delta\varphi) + \frac{1}{2A}(-\Delta\vartheta + A) \int_0^1 \cos(A\tau^2 + (\Delta\vartheta - A)\tau + \Delta\varphi) \cdot d\tau$

$S_2'(A) = \frac{1}{2A}(\sin(A + \Delta\vartheta - A + \Delta\varphi) - \sin(\Delta\varphi)) + \frac{1}{2A}(-\Delta\vartheta + A) \cdot C(A)$

$S_2'(A) = \frac{1}{2A}[\sin(\Delta\vartheta + \Delta\varphi) - \sin(\Delta\varphi) + (A - \Delta\vartheta) \cdot C(A)]$

Second $S_1'(A)$:

$S_1'(A) = \int_0^1 \cos(A\tau^2 + (\Delta\vartheta - A)\tau + \Delta\varphi) \cdot \tau^2 \cdot d\tau$

$S_1'(A) = \frac{1}{2A} \int_0^1 \tau \cdot 2A\tau \cdot \cos(A\tau^2 + (\Delta\vartheta - A)\tau + \Delta\varphi) \cdot d\tau$

$S_1'(A) = \frac{1}{2A} \int_0^1 \tau \cdot (2A\tau + \Delta\vartheta - A - \Delta\vartheta + A) \cdot \cos(A\tau^2 + (\Delta\vartheta - A)\tau + \Delta\varphi) \cdot d\tau$

$S_1'(A) = \frac{1}{2A} \int_0^1 \tau \cdot (2A\tau + \Delta\vartheta - A) \cdot \cos(A\tau^2 + (\Delta\vartheta - A)\tau + \Delta\varphi) \cdot d\tau + \frac{1}{2A} \int_0^1 \tau \cdot (-\Delta\vartheta + A) \cdot \cos(A\tau^2 + (\Delta\vartheta - A)\tau + \Delta\varphi) \cdot d\tau$

$S_1'(A) = \frac{1}{2A} \int_0^1 \tau \cdot (2A\tau + \Delta\vartheta - A) \cdot \cos(A\tau^2 + (\Delta\vartheta - A)\tau + \Delta\varphi) \cdot d\tau + \frac{1}{2A}(-\Delta\vartheta + A) \cdot \int_0^1 \tau \cdot \cos(A\tau^2 + (\Delta\vartheta - A)\tau + \Delta\varphi) \cdot d\tau$

$S_1'(A) = \frac{1}{2A} \int_0^1 \tau \cdot (2A\tau + \Delta\vartheta - A) \cdot \cos(A\tau^2 + (\Delta\vartheta - A)\tau + \Delta\varphi) \cdot d\tau + \frac{1}{2A}(-\Delta\vartheta + A) \cdot S_2'(A)$

$S_1'(A) = \frac{1}{2A}[\int_0^1 \tau \cdot (2A\tau + \Delta\vartheta - A) \cdot \cos(A\tau^2 + (\Delta\vartheta - A)\tau + \Delta\varphi) \cdot d\tau + (A - \Delta\vartheta) \cdot S_2'(A)]$

Integrating in parts

$S_1'(A) = \frac{1}{2A}[\Big|_0^1 \tau \cdot \sin(A\tau^2 + (\Delta\vartheta - A)\tau + \Delta\varphi) - \int_0^1 \sin(A\tau^2 + (\Delta\vartheta - A)\tau + \Delta\varphi) \cdot d\tau + (A - \Delta\vartheta) \cdot S_2'(A)]$

$S_1'(A) = \frac{1}{2A}[\sin(\Delta\vartheta + \Delta\varphi) - S(A) + (A - \Delta\vartheta) \cdot S_2'(A)]$

Solution Algorithmically:

$C = C(A)$

$S = S(A)$

$D = \sin(\Delta\vartheta + \Delta\varphi)$

$E = \sin(\Delta\varphi)$

$F = A - \Delta\vartheta$

Based on these $S_2'(A)$ and $S_1'(A)$:

$S_2'(A) = \frac{1}{2A}(D - E + F \cdot C)$

$S_1'(A) = \frac{1}{2A}(D - S + F \cdot S_2'(A))$

And final result $S'(A)$:

$S'(A) = S_1'(A) - S_2'(A) = \frac{2A \cdot (D - S + F \cdot S_2'(A)) - 2A \cdot (D - E + F \cdot C)}{4A^2}$

$S^{'}(A) = \frac{-2AS+2AF\cdot S_2^{'}(A)+2AE-2AFC)}{4A^2}$
$S^{'}(A) = \frac{-2AS+F(D-E+FC)+2AE-2AFC)}{4A^2}$
$S^{'}(A) = \frac{2A(E-S-FC)+F(D-E+FC)}{4A^2}$

6.5.2 Degenerate Case, A = 0, $\Delta\vartheta \neq 0$

First $S_2^{'}(0)$:

$S_2^{'}(0) = \int_0^1 \cos(\Delta\vartheta\tau + \Delta\varphi)\cdot\tau\cdot d\tau$
$S_2^{'}(0) = \frac{1}{\Delta\vartheta}\int_0^1 \tau\cdot\Delta\vartheta\cdot\cos(\Delta\vartheta\tau + \Delta\varphi)\cdot d\tau$

Integrating in parts:

$S_2^{'}(0) = \frac{1}{\Delta\vartheta}[\Big|_0^1 \tau\cdot\sin(\Delta\vartheta\tau + \Delta\varphi) - \int_0^1 \sin(\Delta\vartheta\tau + \Delta\varphi)\cdot d\tau]$
$S_2^{'}(0) = \frac{1}{\Delta\vartheta}[\Big|_0^1 \tau\cdot\sin(\Delta\vartheta\tau + \Delta\varphi) - \int_0^1 \sin(\Delta\vartheta\tau + \Delta\varphi)\cdot d\tau]$
$S_2^{'}(0) = \frac{1}{\Delta\vartheta}[\sin(\Delta\vartheta + \Delta\varphi) + \frac{1}{\Delta\vartheta}\Big|_0^1 \cos(\Delta\vartheta\tau + \Delta\varphi)\cdot d\tau]$
$S_2^{'}(0) = \frac{1}{\Delta\vartheta}[\sin(\Delta\vartheta + \Delta\varphi) + \frac{1}{\Delta\vartheta}(\cos(\Delta\vartheta + \Delta\varphi) - \cos(\Delta\varphi))]$
$S_2^{'}(0) = \frac{\Delta\vartheta\sin(\Delta\vartheta+\Delta\varphi)+\cos(\Delta\vartheta+\Delta\varphi)-\cos(\Delta\varphi)}{(\Delta\vartheta)^2}$

Second $S_1^{'}(0)$:

$S_1^{'}(0) = \int_0^1 \cos(\Delta\vartheta\tau + \Delta\varphi)\cdot\tau^2\cdot d\tau$
$S_1^{'}(0) = \frac{1}{\Delta\vartheta}\int_0^1 \tau^2\cdot\Delta\vartheta\cdot\cos(\Delta\vartheta\tau + \Delta\varphi)\cdot d\tau$

Integrating in parts:

$S_1^{'}(0) = \frac{1}{\Delta\vartheta}(\Big|_0^1 \tau^2\cdot\sin(\Delta\vartheta\tau + \Delta\varphi) - 2\int_0^1 \tau\sin(\Delta\vartheta\tau + \Delta\varphi)\cdot d\tau)$
$S_1^{'}(0) = \frac{1}{\Delta\vartheta}(\sin(\Delta\vartheta + \Delta\varphi) - 2\int_0^1 \tau\sin(\Delta\vartheta\tau + \Delta\varphi)\cdot d\tau)$
$S_1^{'}(0) = \frac{1}{\Delta\vartheta}(\sin(\Delta\vartheta + \Delta\varphi) - \frac{2}{\Delta\vartheta}\int_0^1 \tau\cdot\Delta\vartheta\sin(\Delta\vartheta\tau + \Delta\varphi)\cdot d\tau)$

Integrating in parts second time:

$S_1^{'}(0) = \frac{1}{\Delta\vartheta}(\sin(\Delta\vartheta + \Delta\varphi) - \frac{2}{\Delta\vartheta}(-\Big|_0^1 \tau\cdot\cos(\Delta\vartheta\tau + \Delta\varphi) + \int_0^1 \cos(\Delta\vartheta\tau + \Delta\varphi)\cdot d\tau)))$
$S_1^{'}(0) = \frac{1}{\Delta\vartheta}(\sin(\Delta\vartheta + \Delta\varphi) - \frac{2}{\Delta\vartheta}(-\cos(\Delta\vartheta + \Delta\varphi) + \frac{1}{\Delta\vartheta}\Big|_0^1 \sin(\Delta\vartheta\tau + \Delta\varphi)\cdot d\tau)))$
$S_1^{'}(0) = \frac{1}{\Delta\vartheta}(\sin(\Delta\vartheta + \Delta\varphi) - \frac{2}{\Delta\vartheta}(-\cos(\Delta\vartheta + \Delta\varphi) + \frac{1}{\Delta\vartheta}(\sin(\Delta\vartheta + \Delta\varphi) - \sin(\Delta\varphi))))$
$S_1^{'}(0) = \frac{(\Delta\vartheta)^2\cdot\sin(\Delta\vartheta+\Delta\varphi)+2\cdot\Delta\vartheta\cos(\Delta\vartheta+\Delta\varphi)-2\cdot\sin(\Delta\vartheta+\Delta\varphi)+2\cdot\sin(\Delta\varphi)}{(\Delta\vartheta)^3}$

And finally:

$S'(0) = S_1^{'}(0) - S_2^{'}(0)$
$S'(0) = \frac{(\Delta\vartheta)^2\cdot\sin(\Delta\vartheta+\Delta\varphi)+2\cdot\Delta\vartheta\cos(\Delta\vartheta+\Delta\varphi)-2\cdot\sin(\Delta\vartheta+\Delta\varphi)+2\cdot\sin(\Delta\varphi)-(\Delta\vartheta)^2\sin(\Delta\vartheta+\Delta\varphi)-\Delta\vartheta\cos(\Delta\vartheta+\Delta\varphi)+\Delta\vartheta\cos(\Delta\varphi)}{(\Delta\vartheta)^3}$
$S'(0) = \frac{\Delta\vartheta\cos(\Delta\vartheta+\Delta\varphi)-2\cdot\sin(\Delta\vartheta+\Delta\varphi)+2\cdot\sin(\Delta\varphi)+\Delta\vartheta\cos(\Delta\varphi)}{(\Delta\vartheta)^3}$

6.5.3 Degenerate Case, A = 0, $\Delta\vartheta$=0

$S^{'}(A) = \int_0^1 \cos(\Delta\varphi)\cdot(\tau^2 - \tau)\cdot d\tau$
$S^{'}(A) = \cos(\Delta\varphi)\int_0^1 (\tau^2 - \tau)\cdot d\tau$

$$S^{'}(A) = \cos(\Delta\varphi)\Big|_0^1 (\tfrac{1}{3}\tau^3 - \tfrac{1}{2}\tau^2) \cdot d\tau$$
$$S^{'}(A) = -\tfrac{1}{6}\cos(\Delta\varphi)$$