Spatial planning with constraints on translational distances between geometric objects *

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Abstract

The main constraint on relative position of geometric objects, used in spatial planning for computing the C-space maps (for example, in robotics, CAD, and packaging), is the relative non-overlapping of objects. This is the simplest constraint in which the minimum translational distance between objects is greater than zero, or more generally, than some positive value. We present a technique, based on the Minkowski operations, for generating the translational C-space maps for spatial planning with more general and more complex constraints on the relative position of geometric objects, such as constraints on various types (not only on the minimum) of the translational distances between objects. The developed technique can also be used, respectively, for spatial planning with constraints on translational distances in a given direction, and rotational distances between geometric objects, as well as for spatial planning with given dynamic geometric situation of moving objects.

Keywords: Spatial planning, Configuration space, Minkowski operations.

1 Introduction

Problems concerning the relative placement of geometric objects are called *spatial planning* problems [29]. Such problems are important in robotics [24], collision detection [21], [27], and computer-aided design and manufacturing (CAD/CAM) [11], [54].

A technique commonly used for solving spatial planning problems is the *configuration space* (or *C-space*) approach, based on representing each placement of an object, i.e., its position and orientation, as a point in some parametric C-space [28], [29]. (Each coordinate of the C-space represents a degree of freedom in the position or orientation of the object.)

Given a collection of objects, the *translational* spatial planning problem consists in computing the set of all the feasible positions (the orientations are fixed) of the objects, where certain constraints on their relative position are specified. The feasible region of (placements of) an object is called the *free* C-space of the object. The prohibited configurations of an object form a *forbidden* region. The *C-space mapping* for a particular spatial planning problem consists in partitioning the C-space into free and forbidden regions, where the latter are called *C-space obstacles*. See [29] and [63] for more details.

*A full version of this paper is available at [42].

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1.1 Previous and related work

Detailed surveys of previous work on spatial planning can be found in [9], [14], [21], [27], [29], and [63]. See also [2], [4], [26], [33], [54], and [58] and references therein for other related works on placement and spatial planning.

The basic constraint on the relative position of geometric objects, used in spatial planning for generating the C-space maps, is the relative non-overlapping of objects; here the minimum translational distance between the objects must be greater than zero, or more generally, than some positive value. Indeed, this is a key requirement in robotics, packaging, and nesting. See, e.g., [4], [26], [27], [53], and [63]. However, the geometric problems arising in design and manufacturing require a placement of objects with more complex constraints on their relative position, such as constraints on the minimum and/or maximum translational distances between objects, and/or their Hausdorff distances. The problem of generating the C-maps with such complex constraints is also interesting theoretically.

Placement problems taking into account the minimum translational distance (MTD) between objects arise in industrial applications, concerning the cutting of materials, the layout of templates on a stock material, and the layout of an IC chip with geometric design constraints. See, e.g., [9], [29], [33], [53], and [54]. In these problems the MTD has to be at least the cutting tolerance of the machine that cuts the shapes out of stock material, or the minimal feasible distances between electronic modules of an IC chip. Placement problems with constraints on the MTD between objects have been formulated in [53] and [54]. The papers [37] and [54] have considered placement problems with consrtaints on the minimal value of MTD between objects, and placement problems with constraints on both the minimal and maximal admissible values of MTD have been considered in [54]. Algorithms for solving various placement problems with constraints on the MTD have been considered in [37], [54], and [56] – [58].

The work in [39] has solved the problem of placement of a pair of objects with constraints on the *minimum* and *maximum* translational distances between them, and has also considered distances involving containment of objects. Placement problems with constraints on several types of *translational* distances between objects, including directional Hausdorff distances between objects, have been studied in [40] and [41]. These works have also studied a Boolean function, called the *geometric situation*, which describes the system of constraints on the relative position of objects, and have formulated and solved placement problems with several other types of geometric situations, namely, with rotational and dynamic geometric situations.

An algorithm for computing the minimum Hausdorff distance between two planar objects under translation is given in [1]. (Efficient computation of Hausdorff distances has applications, e.g., in pattern recognition and computer vision; see [1] and [49].) This work has also studied placement problems taking into account bidirectional Hausdorff distances between objects.

This paper presents a technique, based on the Minkowski operations, for generating the translational C-space maps for spatial planning problems with more general and more complex constraints on the relative position of geometric objects. This technique is an extention of that reported in [40] and [41]. The developed technique can also be used, respectively, for spatial planning with constraints on *translational distances in a given direction*, and/or on *rotational* distances between geometric objects, as well as for spatial planning with given *dynamic* geometric situation of moving objects. A full version of this paper is available at [42].

To formulate the problem let us first present the needed notations and definitions, and then consider the various standard distances between geometric objects. We assume that the geometric objects are *regular* sets (r-sets) in the Euclidean space R^n, for $n = 2$ or 3, i.e., bounded, closed, and semi-analytic subsets of R^n [43]. This means that, for any r-set A of R^n, $A = \mathbf{ki}(A)$, where $\mathbf{i}$ and $\mathbf{k}$ denote the *interior* and the *closure* of sets, respectively [43]. The *complement* and the *boundary* of A are denoted by A^c and ∂A, respectively, and a copy of A translated by a point (or a vector) p is denoted by $A + p$. We denote by A^θ a copy of A rotated by an angle θ about the origin O [24]. The *regularized* set operations on two objects A and B in R^n are defined as $A \otimes^* B = \mathbf{ki}(A \otimes B)$, where $\otimes \in \{\cup, \cap, \setminus\}$; see [43] and [44] for details. The *regularized* complement A^{c^*} of A is defined as $A^{c^*} = \mathbf{ki}(A^c)$. The r-sets are not algebraically closed under the standard set operations, but they are closed under the regularized set operations [43]. For example, the standard intersection $A \cap B$ of r-sets A and B needs not be regular, since its boundary may have *dangling* faces/edges and/or *isolated* points, but $A \cap^* B$ is an r-set.

1.2 Standard distances between geometric objects

Let us consider the following distances between objects A and B (see [1], [19], and [27]):

$$d_1(B, A) = \inf_{a \in A} \inf_{b \in B} \|a - b\|; \quad d_2(B, A) = \sup_{a \in A} \sup_{b \in B} \|a - b\|;$$

$$h(A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\|; \quad h(B, A) = \sup_{b \in B} \inf_{a \in A} \|a - b\|;$$

$$d^*(A, B) = \inf_{a^* \in A^c} \inf_{b \in B} \|a^* - b\|; \quad H(A, B) = \max\{h(A, B), h(B, A)\},$$

where $\|\cdot\|$ is the Euclidean norm. The distance $h(A, B)$ is called the *directed* Hausdorff distance from A to B, and the distance $H(A, B)$ is called the Hausdorff distance between A and B, respectively; see [1]. $H(A, B)$ is a metric on R^n.

These distances between r-sets have the following basic properties [19]:

$$d_1(A, B) > 0 \iff A \cap B = \emptyset; \; d_2(A, A) = \text{diam}(A); \; H(A, B) = 0 \iff A = B.$$

Clearly, $d^*(A, B) = d_1(A^c, B)$. Then we have $d^*(A, B) > 0 \iff B \subset A$.

1.3 Problem formulation

Let A be an unmovable object, and let B be another object, allowed only to translate. Let us consider the following problem:

Problem I For given $\lambda_1, \ldots, \lambda_6$, and the corresponding constraint

$$\begin{aligned} \nu_I(B + p, A) \; = \; & \Big\{ \big[d_1(B + p, A) \leq \lambda_1 \big] \odot_1 \big[d_1(B + p, A) \geq \lambda_2 \big] \Big\} \\ & \odot_2 \Big\{ \big[d_2(B + p, A) \leq \lambda_3 \big] \odot_1 \big[d_2(B + p, A) \geq \lambda_4 \big] \Big\} \\ & \odot_2 \Big\{ \big[d^*(B + p, A) \leq \lambda_5 \big] \odot_1 \big[d^*(B + p, A) \geq \lambda_6 \big] \Big\}, \end{aligned}$$

where $\odot_{1(2)} \in \{\vee, \wedge\}$, find the region $N_I(B, A)$ of all the feasible translations $B + p$ of B, in which $\nu_I(B + p, A)$ holds.

Thus, our goal is to find all the feasible positions p of B with respect to A, under the above constraint on their relative position. This problem is generalization of the well known *Findspace* problem, formulated in [29]. If we let $p = O$, then the function $\nu_I(B, A)$ can be interpreted as the generalized Boolean *distance query*; see [27] for detailes.

2 Preliminaries

2.1 Minkowski operations

The *Minkowski sum*, and the *Minkowski diffference* of objects A and B are defined as

$$A \oplus B = \{a + b \mid a \in A, b \in B\} = \bigcup_{b \in B} (A + b), \text{ and } A \ominus B = \bigcap_{b \in B} (A + b) = (A^c \oplus B)^c,$$

respectively [30], [49]. See Figure 1(a). The Minkowski subtraction is a *dual* operation of the Minkowski addition. Note that $A \oplus B = (A^c \ominus B)^c$.

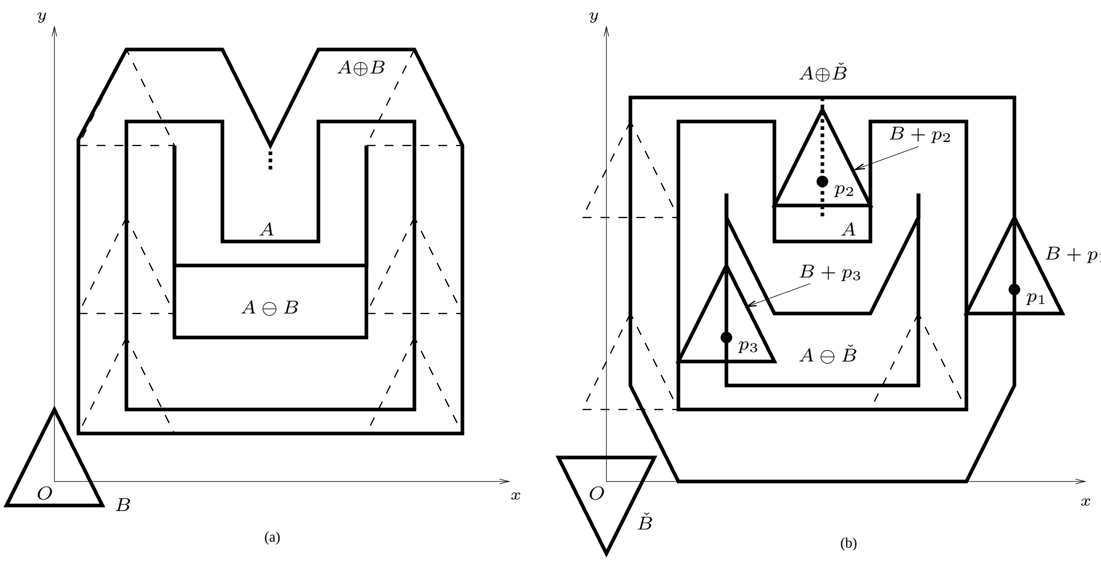


Figure 1: (a) The Minkowski sum $A \oplus B$, and the Minkowski difference $A \ominus B$ of objects A and B. (b) The objects $A \oplus \check{B}$, and $A \ominus \check{B}$. Here $p_1 \in \partial(A \oplus \check{B})$, $p_2 \in \partial(\mathbf{i}A \oplus \mathbf{i}\check{B})$, $(B + p_{1,2}) \dot{\cap} A$, and $p_3 \in \partial(A \ominus \check{B})$, $(B + p_3) \dot{\subset} A$, respectively. (Dashed lines show an objects $B + p$, where (a) $p \in \partial A$, (b) $p \in \partial(\mathbf{i}A \oplus \mathbf{i}\check{B})$ (resp., $p \in \partial(A \ominus \check{B})$). Dotted lines show pieces of $\partial(\mathbf{i}A \oplus \mathbf{i}B)$ (resp., $\partial(\mathbf{i}A \oplus \mathbf{i}\check{B})$).)

Let $\check{B}$ be the *reflection* of B with respect to the origin O, i.e., $\check{B} = \{-b \mid b \in B\}$. (For notational convenience, the object $\check{B}$ is sometimes denoted by $-B$.) Then the *dilation*, and the *erosion* of A by B are defined as

$$A \oplus \check{B} = \{a - b \mid a \in A, b \in B\} = \bigcup_{b \in B} (A - b), \text{ and } A \ominus \check{B} = \bigcap_{b \in B} (A - b) = (A^c \oplus \check{B})^c,$$

respectively. See Figure 1(b).

Many properties of the Minkowski operations are well known and well studied. See, e.g., [3], [14], [19], [26], [30], [31], and [49] for details. In this section we consider the properties of the Minkowski operations that we need for our purpose.

In [8] and [14] it is shown that if an object A is a convex then $A \ominus B = A \ominus CH(B) = A \ominus ext(B)$, where $CH(B)$ denote the *convex hull* of B, and $ext(B)$ denote the set of *extreme points* of B, i.e., the set of vertices of $CH(B)$. In [31] it is shown that the Minkowski sum $A \oplus B$

of two r-sets A and B always results in an r-set, whereas the Minkowski difference $A \ominus B$ could be a non-regular set; see Figure 1(a).

Since $A \oplus (-(B+p)) = (A \oplus \check{B}) - p$, we have (see, e.g., [3], [17], [29], [30], [38], and [55]):

$$\begin{array}{lcl} (B+p) \cap A \neq \emptyset & \iff & p \in A \oplus \check{B}; \\ (B+p) \cap A = \emptyset & \iff & p \in (A \oplus \check{B})^c; \\ (B+p) \dot{\cap} A & \iff & p \in \partial(\mathbf{i}A \oplus \mathbf{i}\check{B}), \end{array} \tag{1}$$

where $B \dot{\cap} A = [(\mathbf{i}A \cap \mathbf{i}B = \emptyset) \wedge (\partial A \cap \partial B \neq \emptyset)]$ denotes the *outer touching* of the objects A and B. The object $A \oplus \check{B}$ is also called the *C-space obstacle* of B relative to A [29]. The set $\partial(\mathbf{i}A \oplus \mathbf{i}\check{B})$ has been introduced in [3], where it is referred to as the *outer envelope* of A and $\check{B}$. In [3] it is shown that $\partial(A \oplus \check{B}) \subseteq \partial(\mathbf{i}A \oplus \mathbf{i}\check{B})$, and that the set $\partial(\mathbf{i}A \oplus \mathbf{i}\check{B})$ may have coincident faces/edges and/or isolated vertices, which are *removed* from the open point set $\mathbf{i}(A \oplus \check{B})$; see Figure 1.

From the relationships $\partial(A \ominus \check{B}) = \partial(A^c \oplus \mathbf{i}\check{B})$ and $A \ominus (-(B+p)) = (A \ominus \check{B}) - p$, it follows that if $A \ominus \check{B} \neq \emptyset$, we get (see [30] and [39]):

$$\begin{array}{lcl} (B+p) \subset A & \iff & p \in A \ominus \check{B}; \\ (B+p) \not\subset A & \iff & p \in (A \ominus \check{B})^c; \\ (B+p) \dot{\subset} A & \iff & p \in \partial(A \ominus \check{B}), \end{array} \tag{2}$$

where $B \dot{\subset} A = [(A^c \cap \mathbf{i}B = \emptyset) \wedge (\partial A \cap \partial B \neq \emptyset)]$ denotes the *inner touching* of A and B. See Figure 1(b).

By observations of [3] and [31] we have $(\mathbf{i}A \oplus \mathbf{i}B) \subseteq \mathbf{i}(A \oplus B)$; $(\mathbf{i}A \ominus \mathbf{i}B) \supseteq \mathbf{i}(A \ominus B)$ and

$$\partial(A \oplus B) \subseteq \partial(\mathbf{i}A \oplus \mathbf{i}B); \;\; \partial(A \ominus B) = \partial(\mathbf{i}A \ominus \mathbf{i}B) = \partial(A^c \oplus \mathbf{i}B);$$

$$A \oplus B = (\mathbf{i}A \oplus \mathbf{i}B) \cup \partial(\mathbf{i}A \oplus \mathbf{i}B); \;\; A \ominus B = \mathbf{i}(A \ominus B) \cup \partial(A \ominus B); \tag{3}$$

$$(A \oplus B)^c = (\mathbf{i}A \oplus \mathbf{i}B)^c \backslash \partial(\mathbf{i}A \oplus \mathbf{i}B); \;\; (A \ominus B)^c = \mathbf{k}[(A \ominus B)^c] \backslash \partial(A \ominus B). \tag{4}$$

From the observations of [19] it follows that, for r-sets, we have

$$A \oplus \mathbf{i}B = \mathbf{i}A \oplus B = \mathbf{i}A \oplus \mathbf{i}B; \; A \oplus B = \mathbf{k}(\mathbf{i}A \oplus \mathbf{i}B); \; A \ominus B = A \ominus \mathbf{i}B = \mathbf{i}A \ominus \mathbf{i}B. \tag{5}$$

(Clearly, for non-regular point sets, the above equalities do not necessarily hold.)

Let us consider the difference $\check{B} \ominus A$. From the properties of the Minkowski difference (see [30]) it follows that $p \in \check{B} \ominus A$ if and only if $p \notin A \oplus \check{B}^c$, and then $A \cap (B^c + p) = \emptyset$, i.e., $A \subset (B+p)$. Thus, $B+p$ covers A if and only if $p \in \check{B} \ominus A$. In other words,

$$\begin{array}{lcl} A \subset (B+p) & \iff & p \in \check{B} \ominus A; \\ A \not\subset (B+p) & \iff & p \in (\check{B} \ominus A)^c; \\ A \dot{\subset} (B+p) & \iff & p \in \partial(\check{B} \ominus A). \end{array} \tag{6}$$

(Note that, by previous observations, $\check{B} \ominus A$, in general, is a non-regular set.)

In case where both A and B are allowed to translate, we have $(A \pm q) \oplus (-(B \pm p)) = (A \oplus \check{B}) \mp p \pm q$, $(A \pm q) \ominus (-(B \pm p)) = (A \ominus \check{B}) \mp p \pm q$, and $(-(B \pm p)) \ominus (A \pm q) = (\check{B} \ominus A) \mp p \pm q$, respectively. Then all the relationships of (1), (2), and (6) can be reformulated to handle this more general form. For instance, $(B+p) \subset (A+q) \iff (p-q) \in A \ominus \check{B}$. (For

$A \oplus \check{B}$ this well known fact (see, e.g., [17] and [49]) has widely been used in [2], [9], and [33], for solving various containment problems.) Clearly, for $p = O$, we have

$$\begin{array}{lll} B \cap A \neq \emptyset & \iff & O \in A \oplus \check{B}; \\ B \subset A & \iff & O \in A \ominus \check{B}; \\ A \subset B & \iff & O \in \check{B} \ominus A, \end{array}$$

which are alternative formulations of the *overlapping* of the objects A and B, of the *containment* of B in A, and of the *covering* of A by B, respectively.

2.2 Distances between geometric objects concerning their outer relative position

The standard minimum distance $d_1(B, A)$ does not take into account the "amount" of intersection between objects A and B, since $d_1(B, A) = 0$, for $A \cap B \neq \emptyset$, regardless of how much they overlap. Minimum translational distance constraints that take into account penetration between objects have been proposed in [5], [7], [35], [37], and [54]. The work in [42] consider one specific set of definitions of such minimum distances, since it has been defined in different ways. In this section we consider the translational distances, as defined in [7] and [37].

The *minimum translational* distance $MTD(A, B)$, introduced in [7], is defined as

$$MTD(A, B) = \begin{cases} -MTD^{+}(A, B), & \text{for } A \cap B \neq \emptyset; \\ MTD^{+}(A, B), & \text{otherwise}, \end{cases}$$

where $MTD^{+}(A, B) = \inf\{\|t\| \mid A \dot{\cap} (B + t)\}$.

The distances $\gamma_{1,2}(B, A)$ between A and B, suggested in [37], are defined as

$$\gamma_1(B, A) = \begin{cases} \inf_{c \in \partial(A \oplus \check{B})} \|c\|, & \text{for } A \cap B = \emptyset; \\ 0, & \text{for } A \dot{\cap} B; \\ -\inf_{c \in \partial(A \oplus \check{B})} \|c\|, & \text{for } A \cap B \neq \emptyset, \end{cases}$$

$$\gamma_2(B, A) = \sup_{c \in \partial(A \oplus \check{B})} \|c\|.$$

See Figure 2. The distance $\gamma_1(B, A)$ is defined by the above relationship only in case where $\partial(A \oplus \check{B}) = \partial(\mathbf{i}A \oplus \mathbf{i}\check{B})$ (see subsection 2.1). Then, in general, we obtain that

$$\gamma_1(B, A) = \begin{cases} \inf_{c \in \partial(\mathbf{i}A \oplus \mathbf{i}\check{B})} \|c\|, & \text{for } A \cap B = \emptyset; \\ -\inf_{c \in \partial(\mathbf{i}A \oplus \mathbf{i}\check{B})} \|c\|, & \text{otherwise}. \end{cases}$$

See Figure 3. By the observations of subsection 2.1, we get $\gamma_1(B, A) = \gamma_1(O, \mathbf{i}A \oplus \mathbf{i}\check{B})$, and

$$\gamma_1(B, A) \begin{cases} < 0, & \text{for } O \in \mathbf{i}A \oplus \mathbf{i}\check{B}; \\ = 0, & \text{for } O \in \partial(\mathbf{i}A \oplus \mathbf{i}\check{B}); \\ > 0, & \text{for } O \in (A \oplus \check{B})^{c}. \end{cases}$$

Note that in the above relationships the set $\mathbf{i}A$ (resp., $\mathbf{i}\check{B}$) can be replaced by A (resp., $\check{B}$), since $\mathbf{i}A \oplus \mathbf{i}\check{B} = A \oplus \mathbf{i}\check{B} = \mathbf{i}A \oplus \check{B}$, for r-sets. See [42] for more details.

It can easily be shown that $\gamma_1(B, A) = MTD(A, B)$. Therefore we denote the minimum translational distance between A and B by $\gamma_1(B, A)$.

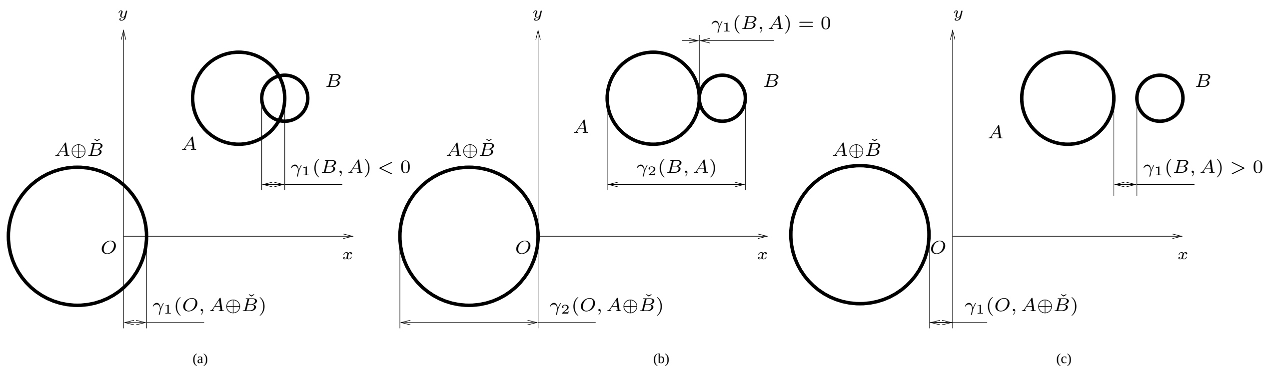


Figure 2: The distances $\gamma_{1,2}(B,A)$, for various relative positions of A and B, in case where $\partial(A\oplus\check{B}) = \partial(\mathbf{i}A\oplus\mathbf{i}\check{B})$. Here (a) $A\cap B\neq\emptyset$, (b) $A\dot{\cap}B$, and (c) $A\cap B=\emptyset$, respectively.

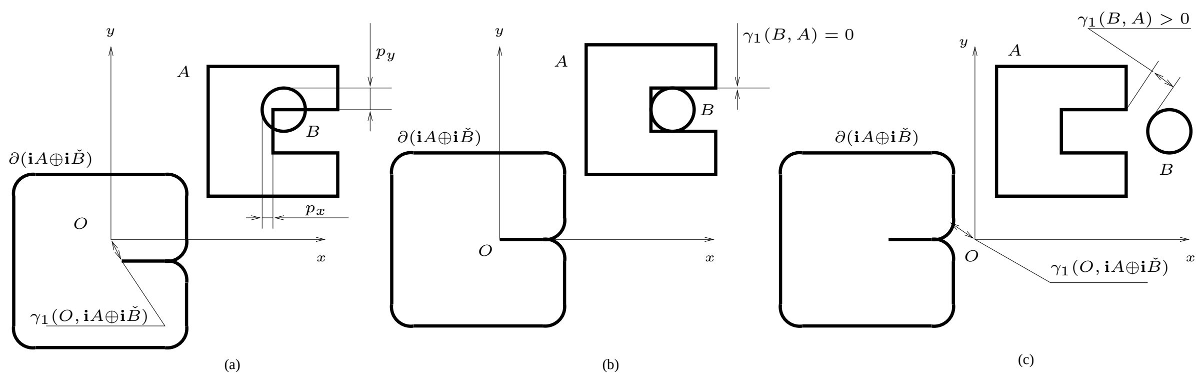


Figure 3: The distance $\gamma_1(B,A)$, for various relative positions of A and B, in case where $\partial(A\oplus\check{B})\subset\partial(\mathbf{i}A\oplus\mathbf{i}\check{B})$. Here (a) $A\cap B\neq\emptyset$; p_x, p_y are the values of minimal translations of B in directions x and y, corresponding to $\gamma_1(B,A)$, (b) $A\dot{\cap}B$, and (c) $A\cap B=\emptyset$, respectively.

The properties of translational distances have been well studied. (See, e.g., [7], [17], [22], [34], [35], [37], [39], [54], and [56]. See also [42] for basic properties of $\gamma_{1,2}(B,A)$.) In [7], [17], [37], and [56] it is shown that $\gamma_1(B,A)$ (resp., $\gamma_2(B,A)$) corresponds to the minimal (resp., maximal) translation $B+p$ of B relative to A that reaches an outer touching $(B+p)\dot{\cap}A$, and that $\gamma_1(B,A)=d_1(B,A)$, for $A\cap B=\emptyset$, and $\gamma_2(B,A)=\gamma_2(O,A\oplus\check{B})=d_2(B,A)$. The distances $\gamma_{1,2}(B,A)$ are invariant with respect to both rotations and translations, i.e., $\gamma_{1,2}(B^\theta+p, A^\theta+p)=\gamma_{1,2}(B,A)$; see [35], [39], [56], and [57].

The papers [37] and [56] have considered the family of surfaces $\partial(A\oplus\check{B}\oplus\lambda K)$, for $\lambda\geq 0$, where λK is the ball of radius λ centered at O. In these works it is shown that $\gamma_1(B+p,A)=\lambda$, for $p\in\partial(A\oplus\check{B}\oplus\lambda K)$. The surfaces $\partial(A\oplus\check{B}\ominus|\lambda|K)$, with similar properties, for negative values of λ, have been defined in [56]. (Note that the above relationship holds only in case where $\partial(A\oplus\check{B}\oplus\lambda K)=\partial(\mathbf{i}A\oplus\mathbf{i}\check{B}\oplus\mathbf{i}\lambda K)$, for $\lambda>0$, and $\partial(A\oplus\check{B})=\partial(\mathbf{i}A\oplus\mathbf{i}\check{B})$, otherwise.)

3 Parametric families of a single object and distances between a point and an object

The parametric family of objects

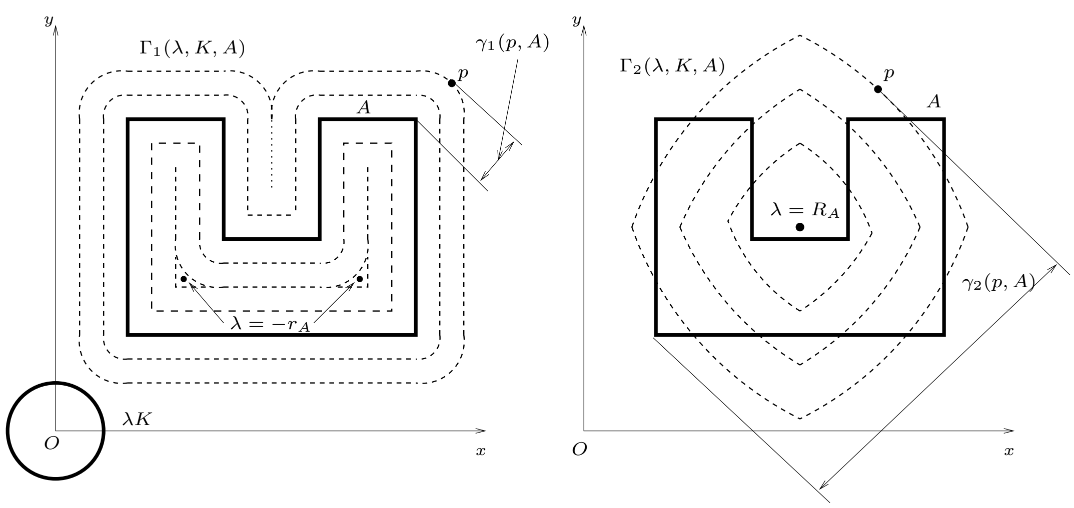
$$\Gamma_1(\lambda, K, A) = \begin{cases} A \oplus \lambda K, & \text{for } \lambda \geq 0; \\ A \ominus |\lambda| K, & \text{for } -r_A \leq \lambda \leq 0, \end{cases}$$

where r_A is the radius of the largest inscribed ball in A, is called the *full parallel pencil* of the object A [19], or, for $\lambda \geq 0$, the *offsets* of A [47]. See Figure 4(a). The object $\Gamma_1(-r_A, K, A)$ is the locus of the centers of all largest inscribed balls in A, and, in general, is a (curved) face/edge. Clearly, $\Gamma_1(-r_A, K, A) \subset A$.

A second parametric family of objects

$$\Gamma_2(\lambda, K, A) = \lambda K \ominus A, \text{ for } \lambda \geq R_A,$$

where R_A is the radius of the smallest circumscribed ball of A, has been introduced in [40] and [41]. See Figure 4(b). From the definition of Minkowski operations, it follows that $\Gamma_1(\lambda, K, A) = \emptyset$ if and only if $\lambda < -r_A$, and $\Gamma_2(\lambda, K, A) = \emptyset$ if and only if $\lambda < R_A$. Since λK is convex, it follows that $\Gamma_2(\lambda, K, A)$ is convex, for any bounded A, and $\Gamma_2(\lambda, K, A) = \Gamma_2[\lambda, K, CH(A)] = \Gamma_2[\lambda, K, ext(A)]$; see subsection 2.1. The object $\Gamma_2(R_A, K, A)$ is a singleton point, which is the center of the (unique) smallest enclosing ball of A. In general, the point of $\Gamma_2(R_A, K, A)$ needs not be in A; see Figure 4(b). From the definitions of $\Gamma_1(\lambda, K, A)$ and $\Gamma_2(\lambda, K, A)$ it follows that $\Gamma_2(\lambda, K, A) \subseteq \Gamma_1(\lambda, K, A)$, for $\lambda \geq R_A$. (An equality holds in case where A is a singleton point.)



(a) (b)

Figure 4: The parametric families of objects $\Gamma_{1,2}(\lambda, K, A)$, for various values of λ, and the distances $\gamma_{1,2}(p, A)$. Here $\Gamma_2(R_A, K, A) \notin A$, dashed curves show $\partial\Gamma_{1,2}(\lambda, K, A)$, and the dotted line shows a piece of $\partial\Gamma_1(\lambda, K, \mathbf{i}A)$. The objects $\Gamma_1(-r_A, K, A)$ and $\Gamma_2(R_A, K, A)$ consist of two singleton points, and a singleton point, respectively.

Let us consider the translational distances $\gamma_{1,2}(p,A)$ between a point p and an object A:

$$\gamma_1(p,A) = \begin{cases} \inf_{a\in\partial A}\|p-a\|, & \text{for } p\in A^c; \\ 0, & \text{for } p\in\partial A; \\ -\inf_{a\in\partial A}\|p-a\|, & \text{for } p\in \mathbf{i}A, \end{cases}$$

$$\gamma_2(p,A) = \sup_{a\in\partial A}\|p-a\|.$$

To lack of space, all proofs here and in the following sections are omitted. The proofs of the observations, lemmas, and theorems are presented in [42].

The distances $\gamma_{1,2}(p,A)$ have the following properties:

Observation 1 *(a) For $\lambda \geq 0$ we have*

$$\gamma_1(p,A)\begin{cases} <\lambda, & \textit{for } p\in\Gamma_1(\lambda,K,\mathbf{i}A); \\ =\lambda, & \textit{for } p\in\partial\Gamma_1(\lambda,K,\mathbf{i}A); \\ >\lambda, & \textit{for } p\in[\Gamma_1(\lambda,K,A)]^c, \end{cases} \quad \gamma_1(p,A)\begin{cases} \leq\lambda, & \textit{for } p\in\Gamma_1(\lambda,K,A); \\ \neq\lambda, & \textit{for } p\in[\partial\Gamma_1(\lambda,K,\mathbf{i}A)]^c; \\ \geq\lambda, & \textit{for } p\in[\Gamma_1(\lambda,K,\mathbf{i}A)]^c. \end{cases}$$

(b) For $-r_A \leq \lambda \leq 0$ we have

$$\gamma_1(p,A)\begin{cases} <\lambda, & \textit{for } p\in\mathbf{i}\Gamma_1(\lambda,K,A); \\ =\lambda, & \textit{for } p\in\partial\Gamma_1(\lambda,K,A); \\ >\lambda, & \textit{for } p\in[\Gamma_1(\lambda,K,A)]^c, \end{cases} \quad \gamma_1(p,A)\begin{cases} \leq\lambda, & \textit{for } p\in\Gamma_1(\lambda,K,A); \\ \neq\lambda, & \textit{for } p\in[\partial\Gamma_1(\lambda,K,A)]^c; \\ \geq\lambda, & \textit{for } p\in\mathbf{k}[\Gamma_1(\lambda,K,A)]^c. \end{cases}$$

Observation 2 *For $\lambda \geq R_A$ we have*

$$\gamma_2(p,A)\begin{cases} <\lambda, & \textit{for } p\in\mathbf{i}\Gamma_2(\lambda,K,A); \\ =\lambda, & \textit{for } p\in\partial\Gamma_2(\lambda,K,A); \\ >\lambda, & \textit{for } p\in[\Gamma_2(\lambda,K,A)]^c, \end{cases} \quad \gamma_2(p,A)\begin{cases} \leq\lambda, & \textit{for } p\in\Gamma_2(\lambda,K,A); \\ \neq\lambda, & \textit{for } p\in[\partial\Gamma_2(\lambda,K,A)]^c; \\ \geq\lambda, & \textit{for } p\in\mathbf{k}[\Gamma_2(\lambda,K,A)]^c. \end{cases}$$

It is clear that $\inf_{p\in R^n}\{\gamma_{1(2)}(p,A)\} = -r_A(R_A)$. Generally, for $i=1,2$, we get

$$\gamma_i(p,A)\begin{cases} \leq\lambda, & \text{for } p\in\Gamma_i(\lambda,K,A); \\ >\lambda, & \text{for } p\in[\Gamma_i(\lambda,K,A)]^c. \end{cases} \tag{7}$$

The topological properties of families $\Gamma_{1,2}(\lambda,K,A)$ have been studied in [42].

4 Correspondence between distances and the parametric families

Let us consider the translational distance $\omega(B,A)$ between the objects A and B. We say that the distance $\omega(B,A)$ *corresponds* to the parametric family of objects $\Omega(\lambda,K,B,A)$ if and only if $\omega(B+p,A)\leq\lambda$, for $p\in\Omega(\lambda,K,B,A)$. (Clearly, $\omega(B+p,A)>\lambda$, for $p\in[\Omega(\lambda,K,B,A)]^c$, and $\omega(B,A)\leq\lambda$, for $O\in\Omega(\lambda,K,B,A)$.) The correspondence between $\omega(B,A)$ and $\Omega(\lambda,K,B,A)$ is denoted by $\omega(B,A)\sim\Omega(\lambda,K,B,A)$.

In special case where an object B is the origin O, the distance function $\omega(B+p,A)$ is reduced to the distance $\omega(O+p,A)=\omega(p,A)$ between a point p and an object A, and the family of objects $\Omega(\lambda,K,B,A)$ is reduced to $\Omega(\lambda,K,O,A)=\Omega(\lambda,K,A)$.

Consider the families of objects $\Gamma_{1,2}(\lambda,K,A)$, and the distances $\gamma_{1,2}(O+p,A)$.

Lemma 3 *For $i = 1, 2$ we have: (a) $\gamma_i(O, A) \sim \Gamma_i(\lambda, K, A)$. (b) $\gamma_i(\pm p, A) = \gamma_i(O, A \mp p)$.*

Follow from the relationship (7), and since $A \oplus \{\pm\check{p}\} = A \mp p$.

Since $\text{diam}(A) = 2R_A$, it can easily be shown that $\gamma_2(p, A) = \text{diam}(\{p\} \cup A) > \lambda$ if and only if $p \in [\Gamma_2(\lambda, K, A)]^c$, for $\lambda \geq 2R_A$. Then we get $\text{diam}(\{O\} \cup A) \sim \Gamma_2(\lambda, K, A)$, for $\lambda \geq 2R_A$. (Note that $\gamma_2(p, A) \leq \text{diam}(\{p\} \cup A) = \text{diam}(A) = \lambda$ if and only if $p \in \Gamma_2(\lambda, K, A)$, for $\lambda \leq 2R_A$.)

The properties of the distances $\gamma_{1,2}(p, A)$ and their corresponding families $\Gamma_{1,2}(\lambda, K, A)$ in case where $A = \bigcup_{j=1}^n A_j$ and/or $A = \bigcap_{j=1}^n A_j$, for $A \neq \emptyset$, have been studied in [42].

In general case where B is a geometric object, but not a single point, we have

Lemma 4 *(a) $\gamma_i(B + p, A) = \gamma_i(B, A - p) = \gamma_i(p, \mathbf{i}A \oplus \mathbf{i}\check{B})$, for $i = 1, 2$. (b) $\gamma_i(p, \mathbf{i}A \oplus \mathbf{i}\check{B}) = \gamma_i[B + \alpha \cdot p, A - (1 - \alpha) \cdot p]$, for $0 \leq \alpha \leq 1$. (c) $\gamma_i(B^\theta \pm p, A^\theta \pm q) = \gamma_i(B^\theta \mp q, A^\theta \mp p) = \gamma_i[\pm p \mp q, (\mathbf{i}A \oplus \mathbf{i}\check{B})^\theta]$, for $i = 1, 2$.*

Remark 1 For $i = 2$, the sets $\mathbf{i}A$ and/or $\mathbf{i}B$ can be replaced by A and/or B, respectively, since $\sup\{\|c\| \mid c \in \partial(\mathbf{i}A \oplus \mathbf{i}\check{B})\} = \sup\{\|c\| \mid c \in \partial(A \oplus \check{B})\}$, i.e., $\gamma_2(p, \mathbf{i}A \oplus \mathbf{i}\check{B}) = \gamma_2(p, A \oplus \check{B})$.

Remark 2 By the first of relationships (5), we have $\gamma_i(B, A) = \gamma_i(B^*, A^*)$, for $i = 1, 2$, where $A^* \in \{A, \mathbf{i}A\}$ and $B^* \in \{B, \mathbf{i}B\}$, respectively.

Let us first consider the parametric family of objects

$$\Gamma_1(\lambda, K, B, A) = \begin{cases} (A \oplus \check{B}) \oplus \lambda K, & \text{for } \lambda \geq 0; \\ (A \oplus \check{B}) \ominus |\lambda| K, & \text{for } -r_{A \oplus \check{B}} \leq \lambda \leq 0, \end{cases}$$

(see Figure 5(a)). Since $\Gamma_1(\lambda, K, B, A) = \Gamma_1(\lambda, K, A \oplus \check{B})$, the object $\Gamma_1(-r_{A \oplus \check{B}}, K, B, A)$ is the locus of the centers of all largest inscribed balls in $A \oplus \check{B}$.

Observation 5 *(a) For $\lambda \geq 0$ we have*

$$\gamma_1(B + p, A) \begin{cases} \leq \lambda, & \textit{for } p \in \Gamma_1(\lambda, K, B, A); \\ < \lambda, & \textit{for } p \in \Gamma_1(\lambda, K, B, \mathbf{i}A); \\ = \lambda, & \textit{for } p \in \partial\Gamma_1(\lambda, K, B, \mathbf{i}A); \\ \geq \lambda, & \textit{for } p \in [\Gamma_1(\lambda, K, B, \mathbf{i}A)]^c. \end{cases}$$

(b) For $-r_{\mathbf{i}A \oplus \mathbf{i}\check{B}} \leq \lambda < 0$ we have

$$\gamma_1(B + p, A) \begin{cases} \leq \lambda, & \textit{for } p \in \Gamma_1(\lambda, \mathbf{i}K, \mathbf{i}B, \mathbf{i}A); \\ < \lambda, & \textit{for } p \in \mathbf{i}\Gamma_1(\lambda, \mathbf{i}K, \mathbf{i}B, \mathbf{i}A); \\ = \lambda, & \textit{for } p \in \partial\Gamma_1(\lambda, \mathbf{i}K, \mathbf{i}B, \mathbf{i}A); \\ \geq \lambda, & \textit{for } p \in \mathbf{k}[\Gamma_1(\lambda, \mathbf{i}K, \mathbf{i}B, \mathbf{i}A)]^c. \end{cases}$$

See Figures 5(a) and 6(a). Hence, the set

$$P(\lambda, K, B, A) = \{p \mid \gamma_1(B + p, A) = \lambda\} = \begin{cases} \partial\Gamma_1(\lambda, K, B, \mathbf{i}A) \text{ for } \lambda \geq 0; \\ \partial\Gamma_1(\lambda, \mathbf{i}K, \mathbf{i}B, \mathbf{i}A), \text{ for } -r_{\mathbf{i}A \oplus \mathbf{i}\check{B}} \leq \lambda < 0, \end{cases}$$

is the surface of the function $\gamma_1(B + p, A)$, and $\inf_{p \in R^n}\{\gamma_1(B + p, A)\} = -r_{\mathbf{i}A \oplus \mathbf{i}\check{B}}$. Thus, we get

Theorem 6 $\gamma_1(B,A) \sim \begin{cases} \Gamma_1(\lambda, K, B, A), \ \textit{for } \lambda \geq 0; \\ \Gamma_1(\lambda, \mathbf{i}K, \mathbf{i}B, \mathbf{i}A), \ \textit{for } -r_{\mathbf{i}A\oplus\mathbf{i}\check{B}} \leq \lambda < 0. \end{cases}$

Note that in case where $\lambda \geq 0$ the Theorem 6 has also been proved in [37] and [56].

Denote by $d_P(A,B)$ the penetration depth of A and B (see [35]). In [42] it is shown that $\gamma_1(B,A) = -d_P(\mathbf{i}A, \mathbf{i}B)$, for $A \cap B \neq \emptyset$. Then we have $d_P(A,B) \sim \Gamma_1(\lambda, K, B, A)$, for $-r_{A\oplus\check{B}} \leq \lambda < 0$.

Let us next consider the parametric family of objects

$$\Gamma_2(\lambda, K, B, A) = \lambda K \ominus (A \oplus \check{B}), \text{ for } \lambda \geq R_{A\oplus\check{B}},$$

proposed in [39] and [40]; see Figures 5(b) and 6(b). Since $\Gamma_2(\lambda, K, B, A) = \Gamma_2(\lambda, K, A\oplus\check{B})$, the object $\Gamma_2(R_{A\oplus\check{B}}, K, B, A)$ is a singleton point, which is the center of the (unique) smallest enclosing ball of $A\oplus\check{B}$. The object $\Gamma_2(\lambda, K, B, A)$ is convex, for any bounded A and B, and $\Gamma_2(\lambda, K, B, A) = \Gamma_2[\lambda, K, CH(A\oplus\check{B})] = \Gamma_2[\lambda, K, ext(A\oplus\check{B})]$.

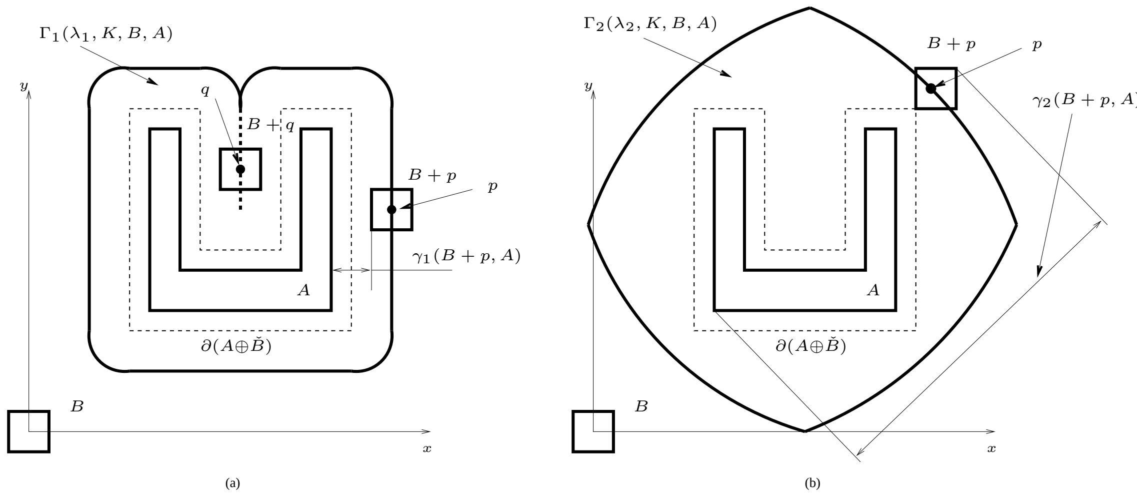


Figure 5: The objects $\Gamma_{1,2}(\lambda_{1,2}, K, B, A)$. Here $\gamma_{1,2}(B+p, A) = \lambda_{1,2}$ and $\lambda_1 > 0$. A dotted line shows a piece of $\partial\Gamma_1(\lambda_1, K, B, \mathbf{i}A)$, and $\gamma_1(B+q, A) = \lambda_1$, for $q \in \partial\Gamma_1(\lambda_1, K, B, \mathbf{i}A)$.

Observation 7 *For* $\lambda \geq R_{A\oplus\check{B}}$ *we have*

$$\gamma_2(B+p, A) \begin{cases} \leq \lambda, \ \textit{for } p \in \Gamma_2(\lambda, K, B, A); \\ < \lambda, \ \textit{for } p \in \mathbf{i}\Gamma_2(\lambda, K, B, A); \\ = \lambda, \ \textit{for } p \in \partial\Gamma_2(\lambda, K, B, A); \\ \geq \lambda, \ \textit{for } p \in \mathbf{k}[\Gamma_2(\lambda, K, B, A)]^c. \end{cases}$$

(Clearly, $\inf_{p\in R^n}\{\gamma_2(B+p, A)\} = R_{A\oplus\check{B}}$.) Then we get the following:

Theorem 8 $\gamma_2(B,A) \sim \Gamma_2(\lambda, K, B, A)$, *for* $\lambda \geq R_{A\oplus\check{B}}$.

By previous observations, we also obtain that $\gamma_2(B+p, A) = \text{diam}((B+p) \cup A) > \lambda$ if and only if $p \in [\Gamma_2(\lambda, K, B, A)]^c$, for $\lambda \geq 2R_{A\oplus\check{B}}$. Hence, $\text{diam}(B \cup A) \sim \Gamma_2(\lambda, K, B, A)$, for $\lambda \geq 2R_{A\oplus\check{B}}$.

The additional (topological and set theoretic) properties of the distances $\gamma_{1,2}(B,A)$ and the families $\Gamma_{1,2}(\lambda, K, B, A)$ have been studied in [42].

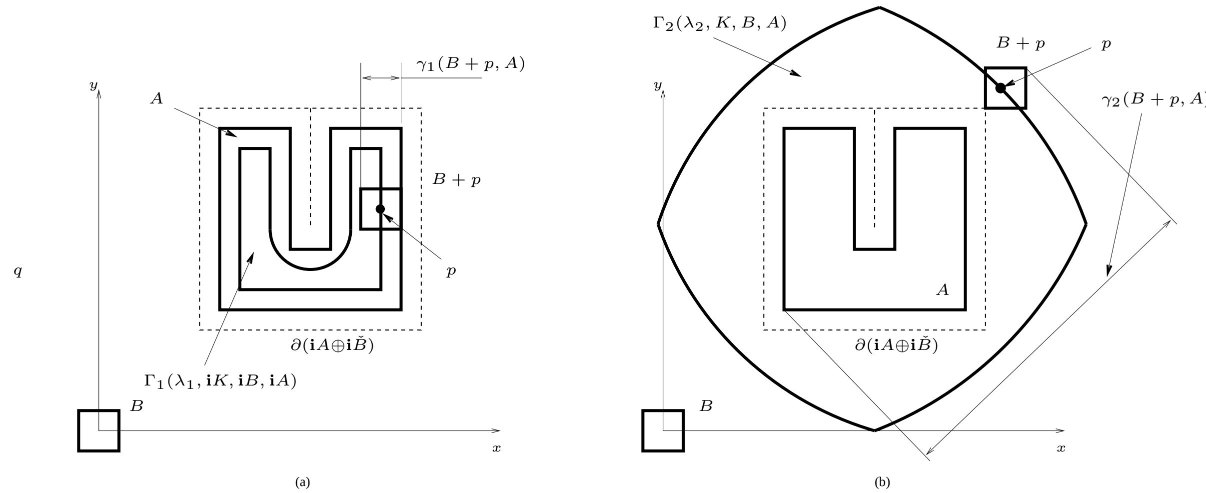


Figure 6: The objects (a) $\Gamma_1(\lambda_1, \mathbf{i}K, \mathbf{i}B, \mathbf{i}A)$ and (b) $\Gamma_2(\lambda_2, K, B, A)$. Here $\gamma_{1,2}(B+p, A) = \lambda_{1,2}$ and $\lambda_1 < 0$. A dashed lines show $\partial(\mathbf{i}A\oplus\mathbf{i}\check{B})$.

5 Distances between geometric objects concerning their inner relative position

The duality of the Minkowski operations provide estimating the *inner* relative position of geometric objects.

5.1 Distances concerning the containment of objects

Let us consider the following translational distances, introduced in [39] and [40]:

$$\eta_1(B,A) = \begin{cases} \inf_{c\in\partial(A\ominus\check{B})} \|c\|, & \text{for } B \not\subset A; \\ 0, & \text{for } B\dot{\subset}A; \\ -\inf_{c\in\partial(A\ominus\check{B})} \|c\|, & \text{for } B\subset A, \end{cases}$$

$$\eta_2(B,A) = \sup_{c\in\partial(A\ominus\check{B})} \|c\|.$$

See Figure 7. (The distances $\eta_{1,2}(B,A)$ are defined only in case where $A\ominus\check{B} \neq \emptyset$.)

The properties of the distances $\eta_{1,2}(B,A)$ have been studied in [39] and [41]. The distance $\eta_1(B,A)$ (resp., $\eta_2(B,A)$) corresponds to the minimal (resp., maximal) translation $B+p$ of B relative to A that reaches an inner touching $(B+p)\dot{\subset}A$. Since $\partial(A\ominus\check{B}) = \partial(A^c\oplus\mathbf{i}\check{B})$, we have $\eta_1(B,A) = -\gamma_1(B,A^c)$ and $\eta_1(B,A) = -d^*(A,B)$, for $B\subset A$.

Lemma 9 *(a)* $\eta_i(p,A) = \gamma_i(p,A)$, *for* $i = 1,2$. *(b)* $\eta_i(B+p,A) = \eta_i(B,A-p) = \eta_i(p,A\ominus\check{B})$. *(c)* $\eta_i[B+\alpha\cdot p, A-(1-\alpha)\cdot p] = \eta_i(p,A\ominus\check{B})$, *for* $0 \le \alpha \le 1$. *(d)* $\eta_i(B^\theta \pm p, A^\theta \pm q) = \eta_i(B^\theta \mp q, A^\theta \mp p) = \eta_i[\pm p \mp q, (A\ominus\check{B})^\theta]$, *for* $i = 1,2$.

It can easily be shown that

$$\eta_1(B,A) \begin{cases} <0, & \text{for } O \in \mathbf{i}(A\ominus\check{B}); \\ =0, & \text{for } O \in \partial(A\ominus\check{B}); \\ >0, & \text{for } O \in (A\ominus\check{B})^c. \end{cases}$$

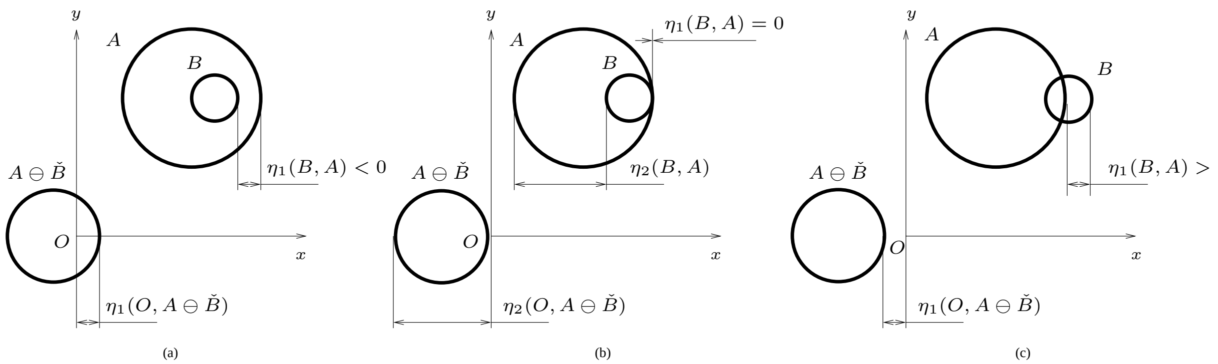


Figure 7: The distances $\eta_{1,2}(B, A)$, for various relative positions of A and B. Here (a) $B\subset A$, (b) $B\dot{\subset} A$, and (c) $B \not\subset A$, respectively.

See Figure 7. The distances $\eta_{1,2}(B, A)$ have used in [40] and [41] to describe the constraints on the relative position of objects in containment problems.

Let us consider the following parametric families of objects [39], [40]:

$$H_1(\lambda, K, B, A) = \begin{cases} (A\ominus\check{B})\oplus\lambda K, & \text{for } \lambda \geq 0; \\ (A\ominus\check{B})\ominus|\lambda|K, & \text{for } -r_{A\ominus\check{B}} \leq \lambda \leq 0, \end{cases}$$

$$H_2(\lambda, K, B, A) = \qquad \lambda K\ominus(A\ominus\check{B}), \qquad \text{for } \lambda \geq R_{A\ominus\check{B}},$$

See Figure 8.

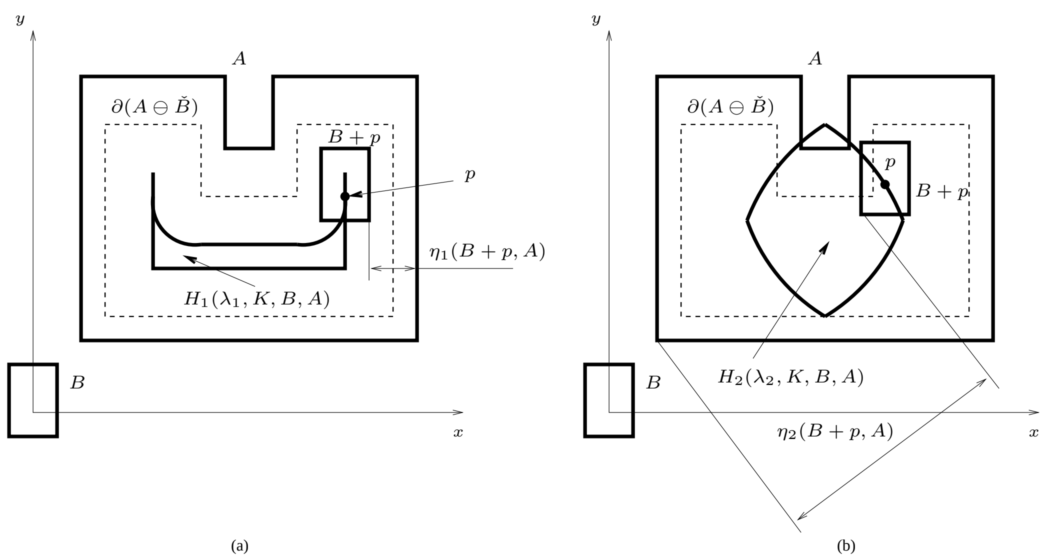


Figure 8: The objects $H_{1,2}(\lambda_{1,2}, K, B, A)$. Here $\eta_{1,2}(B + p, A) = \lambda_{1,2}$ and $\lambda_1 < 0$.

Observation 10 *(a) For $\lambda > 0$ we have*

$$\eta_1(B+p, A) \begin{cases} \leq \lambda, & \textit{for } p \in H_1(\lambda, K, B, A); \\ < \lambda, & \textit{for } p \in H_1(\lambda, \mathbf{i}K, B, A); \\ = \lambda, & \textit{for } p \in \partial H_1(\lambda, \mathbf{i}K, B, A); \\ \geq \lambda, & \textit{for } p \in [H_1(\lambda, \mathbf{i}K, B, A)]^c. \end{cases}$$

(b) For $i = 1, 2$, $-r_{A\ominus\check{B}} \leq \lambda_1 \leq 0$, and $\lambda_2 \geq R_{A\ominus\check{B}}$, respectively, we have

$$\eta_i(B+p, A) \begin{cases} \leq \lambda_i, & \textit{for } p \in H_i(\lambda_i, K, B, A); \\ < \lambda_i, & \textit{for } p \in \mathbf{i}H_i(\lambda_i, K, B, A); \\ = \lambda_i, & \textit{for } p \in \partial H_i(\lambda_i, K, B, A); \\ \geq \lambda_i, & \textit{for } p \in \mathbf{k}[H_i(\lambda_i, K, B, A)]^c. \end{cases}$$

Then $\inf_{p\in R^n}\{\eta_{1(2)}(B+p, A)\} = -r_{A\ominus\check{B}}(R_{A\ominus\check{B}})$. Thus, by above, we obtain the following:

Theorem 11 *$\eta_i(B, A) \sim H_i(\lambda, K, B, A)$, for $i = 1, 2$.*

The additional properties of the distances $\eta_{1,2}(B, A)$ and the families $H_{1,2}(\lambda, K, B, A)$ have been studied in [42].

5.2 Distances concerning the covering of objects

In this section we introduce the distances $\delta_{1,2}(B, A)$ involving the *covering* of A by B:

$$\delta_1(B, A) = \begin{cases} \inf_{c\in\partial(\check{B}\ominus A)} \|c\|, & \text{for } A \not\subset B; \\ 0, & \text{for } A\dot{\subset}B; \\ -\inf_{c\in\partial(\check{B}\ominus A)} \|c\|, & \text{for } A\subset B, \end{cases}$$

$$\delta_2(B, A) = \sup_{c\in\partial(\check{B}\ominus A)} \|c\|.$$

See Figure 9. (The distances $\delta_{1,2}(B, A)$ are defined only in case where $\check{B}\ominus A \neq \emptyset$.)

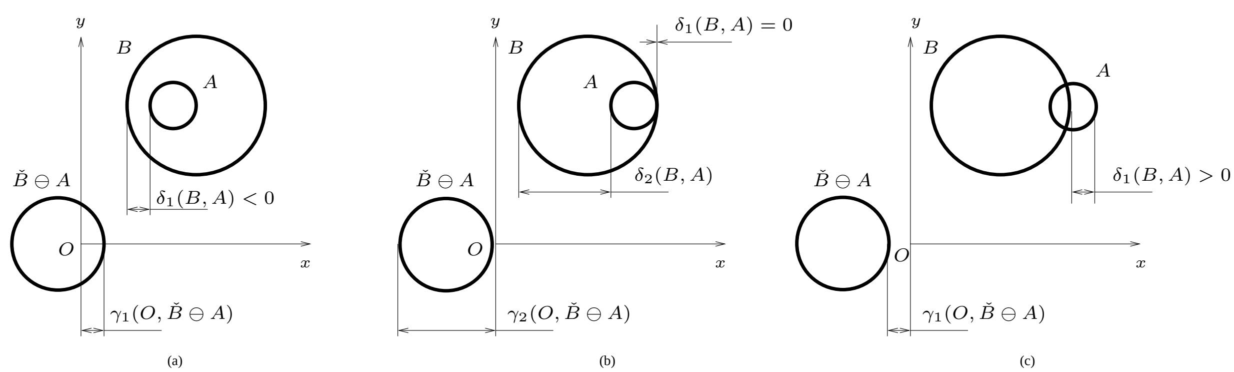


Figure 9: The distances $\delta_{1,2}(B, A)$, for various relative positions of A and B. Here (a) $A\subset B$, (b) $A\dot{\subset}B$, and (c) $A \not\subset B$, respectively.

The properties of the distances $\delta_{1,2}(B, A)$ and $\eta_{1,2}(B, A)$ are similar. The distance $\delta_1(B, A)$ (resp., $\delta_2(B, A)$) corresponds to the minimal (resp., maximal) translation $B + p$ of B relative to A that reaches an inner touching $A\dot{\subset}(B + p)$. The following helpful properties of $\delta_{1,2}(B, A)$ hold: $\delta_1(B, A) = -\gamma_1(B^c, A)$ and $\delta_i(B, A) = \eta_i(\check{A}, \check{B})$, for $i = 1, 2$.

Lemma 12 *(a)* $\delta_i(B+p, A) = \delta_i(B, A-p) = \delta_i(B\ominus\check{A}, -p) = \gamma_i(p, \check{B}\ominus A)$*, for* $i = 1, 2$. *(b)* $\delta_i[B + \alpha\cdot p, A - (1-\alpha)\cdot p] = \delta_i(B\ominus\check{A}, -p)$*, for* $0 \leq \alpha \leq 1$. *(c)* $\delta_i(B^\theta \pm p, A^\theta \pm q) = \delta_i(B^\theta \mp q, A^\theta \mp p) = \delta_i[(B\ominus\check{A})^\theta, \mp p \pm q] = \gamma_i[\pm p \mp q, (\check{B}\ominus A)^\theta]$*, for* $i = 1, 2$.

It can easily be shown that

$$\delta_1(B, A) \begin{cases} < 0, & \text{for } O \in \mathbf{i}(\check{B}\ominus A); \\ = 0, & \text{for } O \in \partial(\check{B}\ominus A); \\ > 0, & \text{for } O \in (\check{B}\ominus A)^c. \end{cases}$$

See Figure 9. The distances $\delta_{1,2}(B, A)$ can be used to formalize the constraints on the relative position of objects in covering problems.

Let us consider the parametric families of objects

$$\Delta_1(\lambda, K, B, A) = \begin{cases} (\check{B}\ominus A)\oplus\lambda K, & \text{for } \lambda \geq 0; \\ (\check{B}\ominus A)\ominus|\lambda|K, & \text{for } -r_{\check{B}\ominus A} \leq \lambda \leq 0, \end{cases}$$

$$\Delta_2(\lambda, K, B, A) = \quad \lambda K\ominus(\check{B}\ominus A), \ \ \text{for } \lambda \geq R_{\check{B}\ominus A}.$$

See Figure 10. Since $\Delta_i(\lambda, K, B, A) = -H_i(\lambda, K, A, B)$, for $i = 1, 2$, we get

Observation 13 *(a) For* $\lambda > 0$ *we have*

$$\delta_1(B+p, A) \begin{cases} \leq \lambda, \ \textit{for } p \in \Delta_1(\lambda, K, B, A); \\ < \lambda, \ \textit{for } p \in \Delta_1(\lambda, \mathbf{i}K, B, A); \\ = \lambda, \ \textit{for } p \in \partial\Delta_1(\lambda, \mathbf{i}K, B, A); \\ \geq \lambda, \ \textit{for } p \in [\Delta_1(\lambda, \mathbf{i}K, B, A)]^c. \end{cases}$$

(b) For $i = 1, 2$, $-r_{\check{B}\ominus A} \leq \lambda_1 \leq 0$*, and* $\lambda_2 \geq R_{\check{B}\ominus A}$*, respectively, we have*

$$\delta_i(B+p, A) \begin{cases} \leq \lambda_i, \ \textit{for } p \in \Delta_i(\lambda_i, K, B, A); \\ < \lambda_i, \ \textit{for } p \in \mathbf{i}\Delta_i(\lambda_i, K, B, A); \\ = \lambda_i, \ \textit{for } p \in \partial\Delta_i(\lambda_i, K, B, A); \\ \geq \lambda_i, \ \textit{for } p \in \mathbf{k}[\Delta_i(\lambda_i, K, B, A)]^c. \end{cases}$$

Thus, $\inf_{p\in R^n}\{\delta_{1(2)}(B+p, A)\} = -r_{\check{B}\ominus A}(R_{\check{B}\ominus A})$. Hence, we can conclude

Theorem 14 $\delta_i(B, A) \sim \Delta_i(\lambda, K, B, A)$*, for* $i = 1, 2$.

Note that the Theorem 14 also follows from the claim (a) of Lemma 12, and from the relationship $\Delta_{1(2)}(\lambda, K, B, A) = \Gamma_{1(2)}(\lambda, K, \check{B}\ominus A)$.

The additional properties of the distances $\delta_{1,2}(B, A)$ and the families $\Delta_{1,2}(\lambda, K, B, A)$ have been studied in [42].

6 Hausdorff distances and corresponding families of objects

Since $H(A, B) = \inf\{\lambda \geq 0 \mid B\subset\Gamma_1(\lambda, K, A), A\subset\Gamma_1(\lambda, K, B)\}$ [19], we have $h(B, A) = \inf\{\lambda \geq 0 \mid B\subset\Gamma_1(\lambda, K, A)\}$. Then $h(B, A) = 0$ in case where $B\subset A$, i.e., the distance $h(B, A)$ does not take into account the "amount" of containment of B in A.

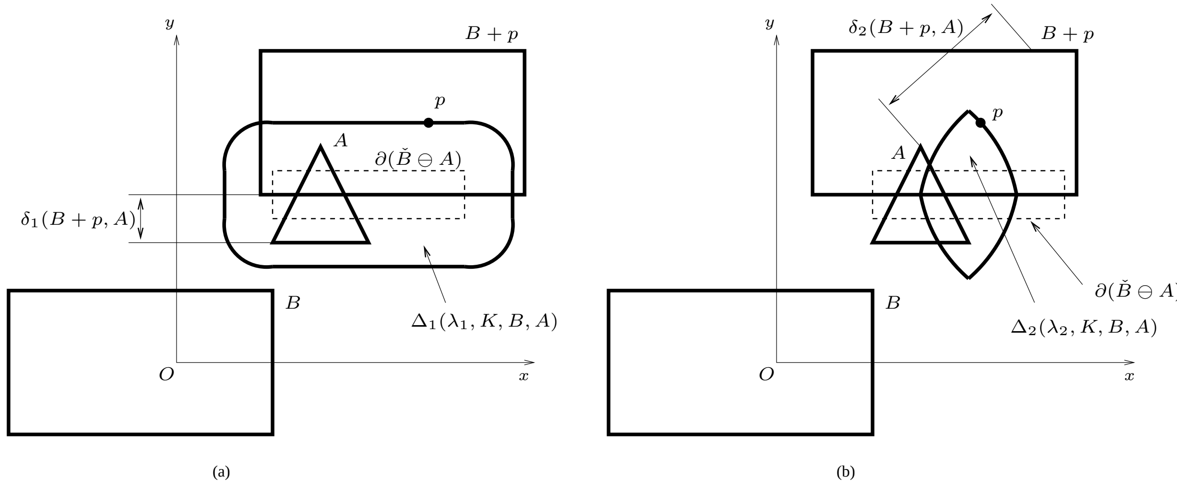


Figure 10: The objects (a) $\Delta_{1,2}(\lambda_{1,2}, K, B, A)$. Here $\delta_{1,2}(B + p, A) = \lambda_{1,2}$ and $\lambda_1 < 0$.

In [41] has been introduced the signed distance $\mu(B, A) = \sup_{b\in B}\{\gamma_1(b, A)\}$, eliminating this shortcoming, and it is shown that

$$\mu(B, A) = \begin{cases} h(B, A), & \text{for } B \not\subset A; \\ \eta_1(B, A), & \text{otherwise.} \end{cases}$$

Then the signed distance $\mu(A, B)$ can be defined as

$$\mu(A, B) = \begin{cases} h(A, B), & \text{for } A \not\subset B; \\ \delta_1(B, A), & \text{otherwise.} \end{cases}$$

Hence, $H(A, B) = \max\{\mu(A, B), \mu(B, A)\}$.

In [1] and [41] has been suggested the parametric family of objects $M_1(\lambda, K, B, A) = \Gamma_1(\lambda, A)\ominus\check{B}$, for $\lambda \geq -r_{A\ominus\check{B}}$, and it is shown that $\mu(B, A) \sim M_1(\lambda, K, B, A)$; see Figure 11(a). The family of objects $M_2(\lambda, K, B, A) = \Gamma_1(\lambda, \check{B})\ominus A$, for $\lambda \geq 0$, has been introduced in [1], and it is shown (in our notation) that $h(A, B) \sim M_2(\lambda, K, B, A)$, and $H(A, B) \sim M_3(\lambda, K, B, A) = [M_1(\lambda, K, B, A) \cap M_2(\lambda, K, B, A)]$, respectively. See Figure 11(b). Clearly, for $-r_{\check{B}\ominus A} \leq \lambda \leq 0$, we have $\mu(A, B) \sim M_2(\lambda, K, B, A)$.

For notational convenience, the distances $\mu(B, A)$, $\mu(A, B)$, and $H(A, B)$ are sometimes denoted by $m_1(B, A)$, $m_2(B, A)$, and $m_3(B, A)$, respectively. Then we get

Theorem 15 *$m_i(B, A) \sim M_i(\lambda, K, B, A)$, for $i = 1 - 3$.*

The objects $M_{1-3}(\lambda, K, B, A)$ have the following simple properties:

$$M_{1(2)}(\lambda, K, B, A) = -M_{2(1)}(\lambda, K, A, B);\ M_3(\lambda, K, B, A) = -M_3(\lambda, K, A, B).$$

Observation 16 *(a) For $i = 1 - 3$, and $\lambda > 0$ we have*

$$m_i(B + p, A) \begin{cases} \leq \lambda, & \textit{for } p \in M_i(\lambda, K, B, A); \\ < \lambda, & \textit{for } p \in \mathbf{i}M_i(\lambda, \mathbf{i}K, \mathbf{i}B, \mathbf{i}A); \\ = \lambda, & \textit{for } p \in M_i(\lambda, K, B, A)\backslash \mathbf{i}M_i(\lambda, \mathbf{i}K, \mathbf{i}B, \mathbf{i}A); \\ \geq \lambda, & \textit{for } p \in \mathbf{k}[M_1(\lambda, \mathbf{i}K, \mathbf{i}B, \mathbf{i}A)]^c. \end{cases}$$

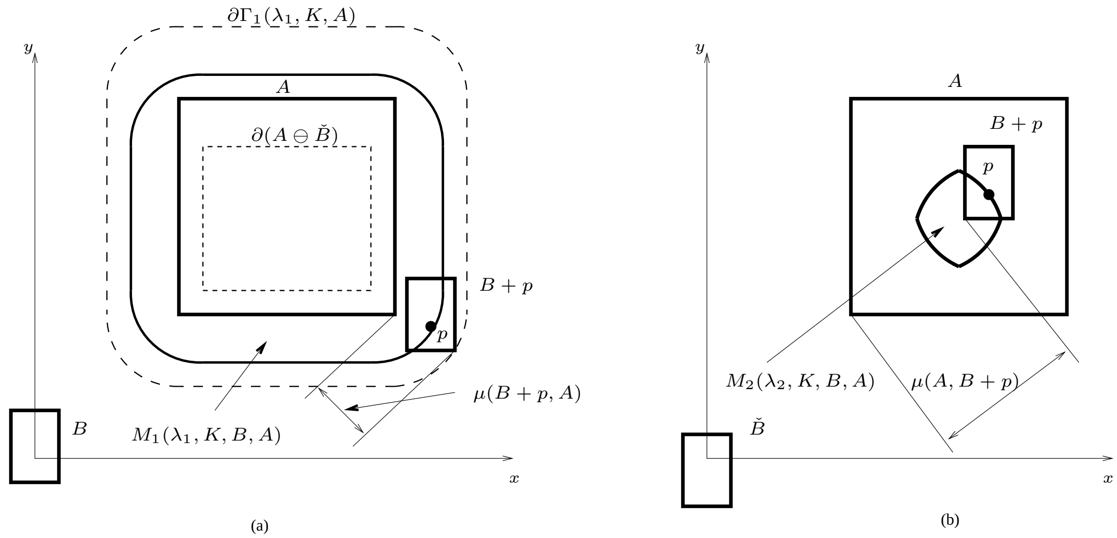


Figure 11: The objects (a) $M_{1,2}(\lambda_{1,2}, K, B, A)$. Here, respectively, $\mu(B + p, A) = \lambda_1 > 0$, $\mu(A, B + p) = \lambda_2 > 0$, and $\partial\Gamma_1(\lambda_1, K, A) = \partial\Gamma_1(\lambda_1, \mathbf{i}K, \mathbf{i}A)$, $\partial\Gamma_1(\lambda_2, K, \check{B}) = \partial\Gamma_1(\lambda_2, \mathbf{i}K, \mathbf{i}\check{B})$.

(b) For $i = 1, 2$, $-r_{A\ominus\check{B}} \leq \lambda_1 \leq 0$, and $-r_{\check{B}\ominus A} \leq \lambda_2 \leq 0$ we have

$$m_i(B + p, A) \begin{cases} \leq \lambda_i, \text{ for } p \in M_i(\lambda_i, K, B, A); \\ < \lambda_i, \text{ for } p \in \mathbf{i}M_i(\lambda_i, K, B, A); \\ = \lambda_i, \text{ for } p \in \partial M_i(\lambda_i, K, B, A); \\ \geq \lambda_i, \text{ for } p \in \mathbf{k}[M_i(\lambda_i, K, B, A)]^c. \end{cases}$$

Remark 3 In [42] it is shown that, in contrast to the distances considered in Sections 2 – 5, the region where the distance $\mu(B + p, A)$ (resp., $\mu(A, B + p)$) is equal to λ may have the non-empty interior.

The additional properties of the Hausdorff distances and their corresponding families have been studied in [42].

7 Translational distances between geometric objects and translational geometric situations

The distances considered in Sections 2 – 6 are referred to as the *translational* distances (or T-distances). Let $\mathcal{TD}(B, A) = \{\gamma_{1,2}(B, A), \eta_{1,2}(B, A), \delta_{1,2}(B, A), m_{1-3}(B, A)\}$ be a collection of the T-distances between B and A, and let $\omega(B, A) \in \mathcal{TD}(B, A)$. The T-distances $\omega(B, A)$ and their corresponding families $\Omega(\lambda, K, B, A)$ have the following properties:

$$\omega(B + p, A) = \omega(B, A - p) = \omega[B + \alpha \cdot p, A - (1 - \alpha) \cdot p], \text{ for } 0 \leq \alpha \leq 1;$$

$$\omega(B^\theta \pm p, A^\theta \pm q) = \omega(B^\theta \mp q, A^\theta \mp p);$$

$$\Omega(\lambda_1, K, B, A) \subseteq \Omega(\lambda_2, K, B, A), \text{ for any bounded } A \text{ and } B, \text{ and } \lambda_1 \leq \lambda_2;$$

$$\Omega[\lambda, K, B + \alpha \cdot p, A - (1 - \alpha) \cdot p] = \Omega(\lambda, K, B, A) - p, \text{ for } 0 \leq \alpha \leq 1;$$

$$\Omega(\lambda, K, B^\theta \pm p, A^\theta \pm q) = \Omega(\lambda, K, B^\theta \mp q, A^\theta \mp p) = [\Omega(\lambda, K, B, A)]^\theta \mp p \pm q.$$

From the last relationship it follows that $\omega(B \pm p, A \pm q) \leq \lambda$, for $\pm p \mp q \in \Omega(\lambda, K, B, A)$.

By the previous observations, we get

$$\begin{array}{lll} (A \cap B = \emptyset) \wedge (B \not\subset A) & \iff & \big[\gamma_1(B,A) > 0\big] \bigwedge \Big\{\big[\eta_1(B,A) > 0\big] \bigvee \big[\mu(B,A) > 0\big]\Big\}; \\ (A \cap B \neq \emptyset) \wedge (A \subset B) & \iff & \big[\gamma_1(B,A) \leq 0\big] \bigwedge \Big\{\big[\delta_1(B,A) \leq 0\big] \bigvee \big[\mu(A,B) \leq 0\big]\Big\}. \end{array}$$

Hence, the various situations of the relative position of objects A and B can be described by the system of constraints on the T-distances between A and B.

Definition 17 *The relationship $\nu(B+p, A) = [\omega(B+p, A) \odot \lambda]$, where $\odot \in \{<, =, >\}$, is called the primitive translational geometric situation (PTGS) of an object B with respect to an object A.*

Let us consider the PTGS $\nu(B+p, A)$ as an event. Then the class of all the possible PTGS's $\mathcal{S}^+(B, A)$ forms an *algebra* $\mathcal{A}_\mathcal{T}$ of events [23]. For $\mathcal{S}^+(B, A)$ permitting the following definitions:

1. The *union* of events: $\nu_1(B+p, A) \vee \nu_2(B+p, A)$.
2. The *intersection* of events: $\nu_1(B+p, A) \wedge \nu_2(B+p, A)$.
3. The *complement* of event: $[\nu(B+p, A)]^c$.
4. The *certain* event I is the union of all the PTGS's in $\mathcal{S}^+(B, A)$.
5. The *impossible* event 0 is an impossible relative position of objects, for given $\nu(B+p, A)$.

The class $\mathcal{S}(B, A)$ of PTGS's, comprising $\mathcal{S}^+(B, A)$ and 0, forms a completely additive Boolean algebra; see [23].

Let us consider the union of PTGS's. (Recall that $\omega(B, A) \in \mathcal{TD}(B, A)$.) Then we get

$$\begin{aligned} [\omega(B+p, A) \leq \lambda] &= [\omega(B+p, A) < \lambda] \vee [\omega(B+p, A) = \lambda]; \\ [\omega(B+p, A) \neq \lambda] &= [\omega(B+p, A) < \lambda] \vee [\omega(B+p, A) > \lambda]; \\ [\omega(B+p, A) \geq \lambda] &= [\omega(B+p, A) > \lambda] \vee [\omega(B+p, A) = \lambda]. \end{aligned} \tag{8}$$

(Clearly, $[\omega(B+p, A) < \lambda] \wedge [\omega(B+p, A) > \lambda] = 0$.) We define also the TGS of type $[\omega(B+p, A) \to \min]$, since in Sections 4 – 6 have been obtained the possible minimal values of the T-distances. We next add these TGS's to the set of PTGS's:

Definition 18 *The relationship $\nu(B+p, A) = [\omega(B+p, A) \odot \lambda$ or $\omega(B+p, A) \to \min]$, where $\odot \in \{<, \leq, =, \neq, \geq, >\}$, is called the basic translational geometric situation (BTGS) of B relative to A.*

Since $[\omega(B+p, A) < \lambda]^c = [\omega(B+p, A) \geq \lambda]$, $[\omega(B+p, A) = \lambda]^c = [\omega(B+p, A) \neq \lambda]$, and $[\omega(B+p, A) > \lambda]^c = [\omega(B+p, A) \leq \lambda]$, then, by the relationships of (8), and, by DeMorgan's laws, we obtain the following useful relationships:

$$\begin{aligned} [\omega(B+p, A) < \lambda] &= [\omega(B+p, A) \leq \lambda] \wedge [\omega(B+p, A) \neq \lambda]; \\ [\omega(B+p, A) = \lambda] &= [\omega(B+p, A) \leq \lambda] \wedge [\omega(B+p, A) \geq \lambda]; \\ [\omega(B+p, A) > \lambda] &= [\omega(B+p, A) \geq \lambda] \wedge [\omega(B+p, A) \neq \lambda]. \end{aligned}$$

Definition 19 *The Boolean function* $\nu(B+p,A) = f[\nu_1(B+p,A),\ldots,\nu_k(B+p,A)]$ *of BTGS's* $\nu_i(B+p,A)$*, for* $i = 1,\ldots,k$*, is called the translational geometric situation (TGS) of* B *with respect to* A*.*

A particular TGS describes the constraints on the relative position of objects, and can be used as the *objective function* in the spatial planning problems.

8 Constructing the feasible region of an object for given translational geometric situation

Let $\Omega(\lambda,K,B,A)$ is the corresponding family of the distance $\omega(B,A)$. From the observations of Sections 4 – 6, and from the Definition 18 it follows that, for given BTGS $\nu(B+p,A)$, all the feasible translations $B+p$ of B with respect to A are obtained if and only if p belongs to the region $N(B,A)$, where $\nu(B+p,A)$ is true. For example,

$$\begin{aligned} \nu_1(B+p,A) &= [\omega(B+p,A) \le \lambda_1] \iff p \in N_1(B,A) = \Omega(\lambda_1,K,B,A); \\ \nu_2(B+p,A) &= [\omega(B+p,A) > \lambda_2] \iff p \in N_2(B,A) = [\Omega(\lambda_2,K,B,A)]^c. \end{aligned}$$

The correspondence between $\nu(B+p,A)$ and $N(B,A)$ is denoted by $\nu(B+p,A) \sim N(B,A)$.

Consider the TGS $\nu(B+p,A) = \nu_1(B+p,A) \wedge \nu_2(B+p,A)$, where the BTGS's $\nu_{1,2}(B+p,A)$ are as above. Clearly, its corresponding region is $N(B,A) = N_1(B,A) \cap N_2(B,A)$. In case where $\lambda_1 \le \lambda_2$, we have $N(B,A) = \emptyset$, and therefore $\nu(B+p,A)$ is an impossible TGS, i.e., $\nu(B+p,A) = 0$. For the TGS $\nu(B+p,A) = \nu_1(B+p,A) \vee \nu_2(B+p,A)$ we have $N(B,A) = N_1(B,A) \cup N_2(B,A)$. If $\lambda_1 \ge \lambda_2$ then $N(B,A) = R^n$. (Note that in this case $\nu(B+p,A) \ne I$.)

Consider next the TGS $\nu(B+p,A) = [\omega(B+p,A) \le \lambda_1] \vee [\omega(B+p,A) \le \lambda_2]$, where $\lambda_1 \le \lambda_2$. Then $N(B,A) = \Omega(\lambda_1,K,B,A) \cup \Omega(\lambda_2,K,B,A)$ and $\Omega(\lambda_1,K,B,A) \subseteq \Omega(\lambda_2,K,B,A)$, for $\lambda_1 \le \lambda_2$. Hence, $N(B,A) = \Omega(\lambda_2,K,B,A)$. That is, the TGS $\nu(B+p,A)$ corresponding to the region $N(B,A)$ can be represented in different ways, e.g., as $\nu(B+p,A) = [\omega(B+p,A) \le \lambda_2]$ or $\nu(B+p,A) = \bigvee_{i=1}^{n}[\omega(B+p,A) \le \lambda_i]$, where $\lambda_i \le \lambda_2$, for $i = 1,\ldots,n$. Thus, in general, the TGS $\nu(B+p,A)$ does not unique relative to its corresponding region $N(B,A)$. However, for given $N(B,A)$, the unique (minimal) corresponding TGS $\nu(B+p,A)$ can be constructed. Finally, if in the above example, we let $N(B,A) = \Omega(\lambda,K,B,A) \cup \Omega(\lambda_2,K,B,A)$, where $\lambda_1 < \lambda \le \lambda_2$, then we obtain that the TGS $[\omega(B+p,A) \le \lambda_1]$ does not corresponds to the region $\Omega(\lambda,K,B,A)$, however $\nu(B+p,A) \sim N(B,A)$ in this case. Therefore, we get the following:

Proposition 20 *(a) The TGS* $\nu(B+p,A) = f[\nu_1(B+p,A),\ldots,\nu_k(B+p,A)]$ *corresponds to the region* $N(B,A) = F[N_1(B,A),\ldots,N_k(B,A)]$ *if (but not only if)* $\nu_i(B+p,A) \sim N_i(B,A)$*, for* $i = 1,\ldots,k$*, and the operations of union, intersection, and complement of the BTGS's* $\nu_i(B+p,A)$ *of* $\nu(B+p,A)$ *correspond to the operations of union, intersection, and complement of regions* $N_i(B,A)$ *of* $N(B,A)$*. (b)* $\nu(B+p,A)$ *is an impossible TGS if and only if* $N(B,A) = \emptyset$*. (c)* $\nu(B+p,A)$ *is a certain TGS only if (but not if)* $N(B,A) = R^n$*.*

The region $N(B,A)$ is called the *solution* of the TGS $\nu(B+p,A)$. (Note that the region $N_i(B,A)$ of $N(B,A)$ is solution of the BTGS $\nu_i(B+p,A)$, for $i = 1,\ldots,k$.)

Remark 4 Let $\nu(B+p,A)$ be the BTGS concerning the T-distance $\omega(B+p,A)$, and let $\omega(B+p,A) \sim \Omega(\lambda,K,B,A)$, and $\nu(B+p,A) \sim N(B,A)$, respectively. In the next paragraph we assume, for simplicity, that the region $N(B,A)$ does not have coincident faces/edges and/or isolated points, which are removed from the interiors of the objects of $\Omega(\lambda,K,B,A)$, for any λ. Then we get

$$\omega(B+p,A)\begin{cases} \leq \lambda, \text{ for } p \in \Omega(\lambda,K,B,A); \\ < \lambda, \text{ for } p \in \mathbf{i}\Omega(\lambda,K,B,A); \\ = \lambda, \text{ for } p \in \partial\Omega(\lambda,K,B,A); \\ \geq \lambda, \text{ for } p \in \mathbf{k}[\Omega(\lambda,K,B,A)]^c. \end{cases}$$

In this special case the set $\partial\Omega(\lambda,K,B,A)$ is the surface of the function $\omega(B+p,A)$.

The types of objects satisfying the above assumption are sufficiently wide. For example, the convex objects, the polygons/polytops and/or curved objects in general position, i.e., which do not have parallel edges/faces. Thus, one can to construct the solution of a particular TGS according to the more simpler relationships than that is done in Sections 4 – 6. See [42] for details.

Examples. Consider the TGS's $\nu_l(B+p,A)$ and their solutions $N_l(B,A)$, denoted by ν_l, N_l, for short. Denote also $\omega(B+p,A)$, $\Omega(\lambda,K,B,A)$ by ω, $\Omega(\lambda)$. Below we let $i(j)=1,2$.

$$\mathbf{1} \quad \nu_1 = (\gamma_i \leq \lambda_i) \wedge (\gamma_j \geq \lambda_j); \quad N_1 = \Gamma_i(\lambda_i) \cap \mathbf{k}[\Gamma_j(\lambda_j)]^c = \Gamma_i(\lambda_i)\backslash \mathbf{i}\Gamma_j(\lambda_j).$$

$$\mathbf{1a} \quad \nu_{1a} = (\gamma_i < \lambda_i) \wedge (\gamma_j > \lambda_j); \quad N_{1a} = \mathbf{i}\Gamma_i(\lambda_i) \cap [\Gamma_j(\lambda_j)]^c = \mathbf{i}\Gamma_i(\lambda_i)\backslash \Gamma_j(\lambda_j).$$

$$\mathbf{1b} \quad \nu_{1b} = (\gamma_i \leq \lambda_i) \wedge (\gamma_j > \lambda_j); \quad N_{1b} = \Gamma_i(\lambda_i) \cap [\Gamma_j(\lambda_j)]^c = \Gamma_i(\lambda_i)\backslash \Gamma_j(\lambda_j).$$

$$\mathbf{1c} \quad \nu_{1c} = (\gamma_i \geq \lambda_i) \wedge (\gamma_j \geq \lambda_j); \quad N_{1c} = \mathbf{k}[\Gamma_i(\lambda_i)]^c \cap \mathbf{k}[\Gamma_j(\lambda_j)]^c = \mathbf{k}[\Gamma_i(\lambda_i) \cup \Gamma_j(\lambda_j)]^c.$$

$$\mathbf{2} \quad \nu_2 = (\gamma_i = \lambda_i) \wedge (\gamma_j = \lambda_j); \quad N_2 = \partial\Gamma_i(\lambda_i) \cap \partial\Gamma_j(\lambda_j).$$

$$\mathbf{3} \quad \nu_3 = (\gamma_1 \geq \lambda_1) \wedge (\gamma_2 \to \min), \text{ where } \lambda_1 \neq r_{(A\oplus\check{B})};$$

$$N_3 = \mathbf{k}[\Gamma_1(\lambda_1)]^c \cap \Gamma_2(R_{A\oplus\check{B}}) = \Gamma_2(R_{A\oplus\check{B}})\backslash \mathbf{i}\Gamma_1(\lambda_1).$$

$$\mathbf{4} \quad \nu_4 = (\eta_1 \to \min); \quad N_4 = H_1(-r_{A\ominus\check{B}}).$$

$$\mathbf{5} \quad \nu_5 = (\eta_1 \leq \lambda_1 \leq 0) \wedge (\eta_2 \to \min); \quad N_5 = H_1(\lambda_1) \cap H_2(R_{A\ominus\check{B}}).$$

$$\mathbf{6} \quad \nu_6 = [(\gamma_1 = \lambda_1) \wedge (\gamma_2 = \lambda_{\min})] \neq 0; \quad N_6 = \partial\Gamma_1(\lambda_1) \cap \partial\Gamma_2(\lambda_{\min}),$$

$$\text{where } \lambda_{\min} = \begin{cases} \inf\{\lambda \mid \Gamma_2(\lambda)\dot{\subset}\Gamma_1(\lambda_1)\}, \text{ for } \Gamma_2(R_{A\oplus\check{B}}) \in \Gamma_1(\lambda_1); \\ \inf\{\lambda \mid \Gamma_2(\lambda)\dot{\cap}\Gamma_1(\lambda_1)\}, \text{ for } \Gamma_2(R_{A\oplus\check{B}}) \notin \Gamma_1(\lambda_1). \end{cases}$$

See Figure 12. The analysis of the Examples 1 – 6 can be found in [42].

Remark 5 The region $N(B,A)$ may have various topology. It can be open or closed, regular or non-regular, bounded or unbounded, connected or disconnected. So, in the above examples the region N_1 is closed bounded, whereas the region N_{1a} is open bounded; the region N_{1c} is closed unbounded. In case where $N(B,A)$ contains the subset of its boundary, it is neither open nor closed. Thus, in general, $N(B,A)$ is an object with *non-manifold* boundary. The region N_{1b} give an example of such an object. In case where $i=j$, $\lambda_i > \lambda_j$, and $\lambda_{i,j} > 0$, respectively, the object N_1 is regular, i.e.,$N_1 = \Gamma_i(\lambda_i)\backslash^*\Gamma_j(\lambda_j)$. (Note that in case where $\lambda_i = \lambda_j$ we get $N_1 = \partial\Gamma_i(\lambda_i)$, i.e., the object N_1 is non-regular.)

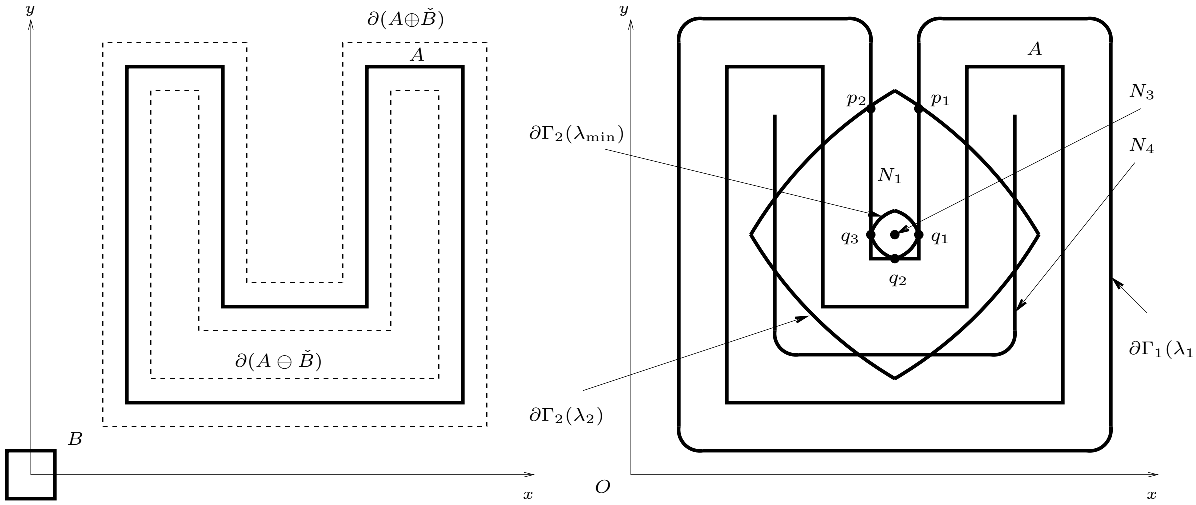


Figure 12: The corresponding regions N_l of TGS's ν_l, for $l = 1, \ldots, 6$, in case where $\Gamma_2(R_{A\oplus\check{B}}) \not\subseteq \Gamma_1(\lambda_1)$. Here $i = 2$, $j = 1$, and $\lambda_1 < \lambda_2$, respectively. The region N_1 is simply connected, $N_2 = \{p_1, p_2\}$, and $N_3 = \Gamma_2(R_{A\oplus\check{B}})$, respectively. The region N_4 is a curve, $N_5 = \emptyset$, and $N_6 = \{q_1, q_2, q_3\}$.

Let us next consider the following problem:

Problem II Find the region N_{II}, corresponding to the TGS

$$\nu_{II} = \big[(\lambda_1 \le \gamma_1 \le \lambda_2) \wedge (\lambda_3 \le \gamma_2 \le \lambda_4)\big] \vee \big[(\lambda_5 \le \eta_1 \le \lambda_6) \wedge (\lambda_7 \le \eta_2 \le \lambda_8)\big].$$

By Example 1, the solution is the region

$$N_{II} = \Big\{\big[\Gamma_1(\lambda_2)\backslash \mathbf{i}\Gamma_1(\lambda_1)\big] \cap \big[\Gamma_2(\lambda_4)\backslash \mathbf{i}\Gamma_2(\lambda_3)\big]\Big\} \cup \Big\{\big[H_1(\lambda_6)\backslash \mathbf{i}H_1(\lambda_5)\big] \cap \big[H_2(\lambda_8)\backslash \mathbf{i}H_2(\lambda_7)\big]\Big\}.$$

See Figure 13. Note that in case where $\lambda_1 > \lambda_2$ or $\lambda_3 > \lambda_4$, and $\lambda_5 > \lambda_6$ or $\lambda_7 > \lambda_8$ we have $N_{II} = \emptyset$ and $\nu_{II} = 0$.

Let us turn to solve the problem, formulated in subsection 1.3. The following regions N_I', for $\odot_1 = \wedge$, $\odot_2 = \vee$, and N_I'', for $\odot_1 = \vee$, $\odot_2 = \wedge$, are the solutions of the Problem I:

$$N_I' = \big[\Gamma_1(\lambda_1)\backslash \mathbf{i}\Gamma_1(\lambda_2)\big] \cup \big[\Gamma_2(\lambda_3)\backslash \mathbf{i}\Gamma_2(\lambda_4)\big] \cup \big[H_1(\lambda_5)\backslash \mathbf{i}H_1(\lambda_6)\big];$$

$$N_I'' = \big[\Gamma_1(\lambda_1) \cup \mathbf{k}[\Gamma_1(\lambda_2)]^c\big] \cap \big[\Gamma_2(\lambda_3) \cup \mathbf{k}[\Gamma_2(\lambda_4)]^c\big] \cap \big[H_1(\lambda_5) \cup \mathbf{k}[H_1(\lambda_6)]^c\big].$$

Clearly, if $\lambda_1 < \lambda_2$, $\lambda_3 < \lambda_4$, and $\lambda_5 < \lambda_6$, then $N_I' = \emptyset$ and $\nu_I' = 0$. In case where $\lambda_1 \ge \lambda_2$, $\lambda_3 \ge \lambda_4$, and $\lambda_5 \ge \lambda_6$ we have $N_I'' = R^d$. (However, $\nu_I'' \ne I$ in this case.)

The additional properties of the TGS's have been studied in [42].

9 Applications

In this section we consider the spatial planning problems with more general and more complex constraints on the distances between geometric objects. We also briefly consider the several other types of geometric situations: the translational geometric situation in a given direction, the rotational, and the dynamic geometric situations, respectively.

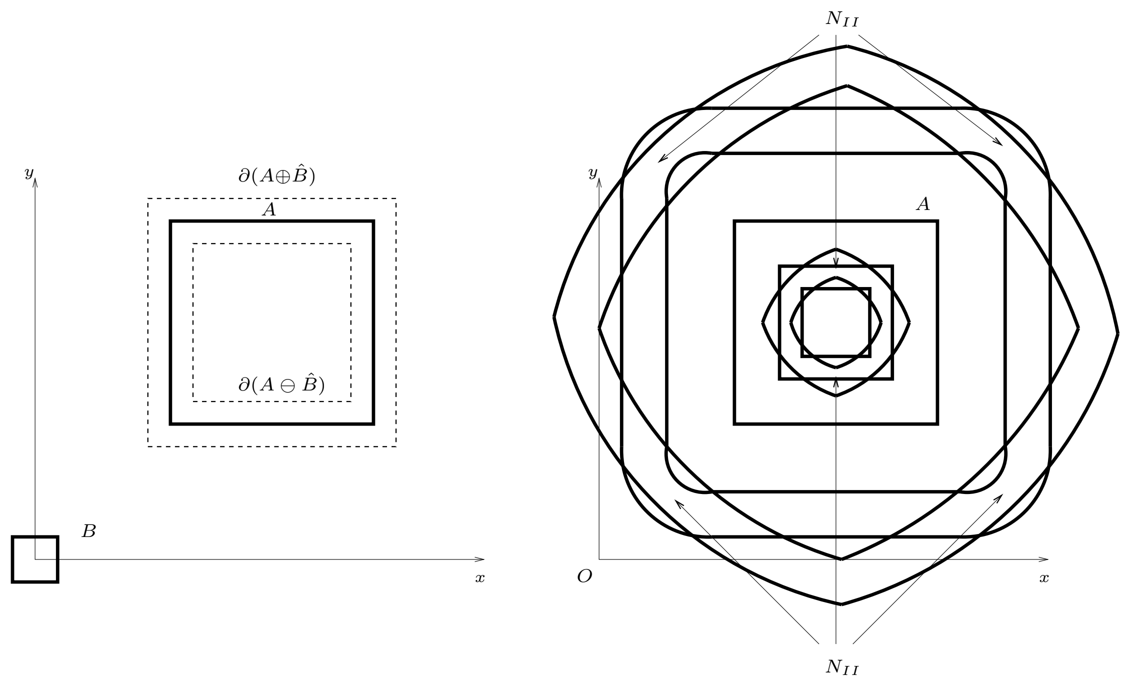


Figure 13: Illustration for the Problem II. Here the region N_{II} is 5-connected.

9.1 Findspace problem

Let $\mathcal{A} = \{A_1, \ldots, A_n\}$ be a collection of n, possibly intersecting, obstacles A_i, completely contained in the region R, and let B be the object moving relative to $\mathcal{A}$ under translations. See Figure 14(a). (For notational convenience, we also denote $\mathcal{A} = \bigcup_{i=1}^{n} A_i$.)

The *translational* Findspace problem [29] is to find all the possible translations $(B+p)\subset R$, such that $(B+p) \cap \mathcal{A} = \emptyset$. In this case p is called the *free* position. (If $(B+p)\dot{\cap}\mathcal{A}$, then p is called the *semi-free* position [4].)

The C-space obstacle of B relative to $\mathcal{A}$ is defined as $CO_B(\mathcal{A}) = \{p \mid (B+p) \cap \mathcal{A} \neq \emptyset\} = \mathcal{A}\oplus\check{B} = \bigcup_{i=1}^{n} CO_B(A_i) = \bigcup_{i=1}^{n}(A_i\oplus\check{B})$ [29]. (The object $[CO_B(\mathcal{A})]^c$ is called the *free C-space* of B relative to $\mathcal{A}$.) The C-space *interior* of B relative to the region R is defined as $CI_B(R) = \{p \mid (B+p)\subseteq R\} = [CO_B(R^c)]^c$; see Figure 14(a). By the definition of the Minkowski difference, we get $CI_B(R) = R\ominus\check{B}$.

In [14] it is shown that the set of all the feasible positions of B relative to $\mathcal{A}$ and R can be represented as $FP_B(\mathcal{A}, R) = (R\backslash\mathcal{A})\ominus\check{B} = (R\ominus\check{B})\backslash(\mathcal{A}\oplus\check{B}) = CI_B(R)\backslash CO_B(\mathcal{A})$; see Figure 14(a).

Let us consider the Findspace problem in terms of the translational geometric situations (TGS), studied in Sections 7 and 8. The conditions $(B+p)\subset R$ and $(B+p) \cap \mathcal{A} = \emptyset$ can be formalized as $[\eta_1(B+p, R) \leq 0]$ and $[\gamma_1(B+p, \mathcal{A}) > 0]$, respectively. In [42] it is shown that $[\gamma_1(B+p, \mathcal{A}) > 0] = \bigwedge_{i=1}^{n}[\gamma_1(B+p, A_i) > 0]$. Then the Findspace problem can be reformulated as follows: Find the region $N(B, \mathcal{A}, R)$ corresponding to the TGS

$$\nu(B+p, \mathcal{A}, R) = \Big[\eta_1(B+p, R) \leq 0\Big] \bigwedge \Big\{ \bigwedge_{i=1}^{n} \Big[\gamma_1(B+p, A_i) > 0\Big] \Big\}.$$

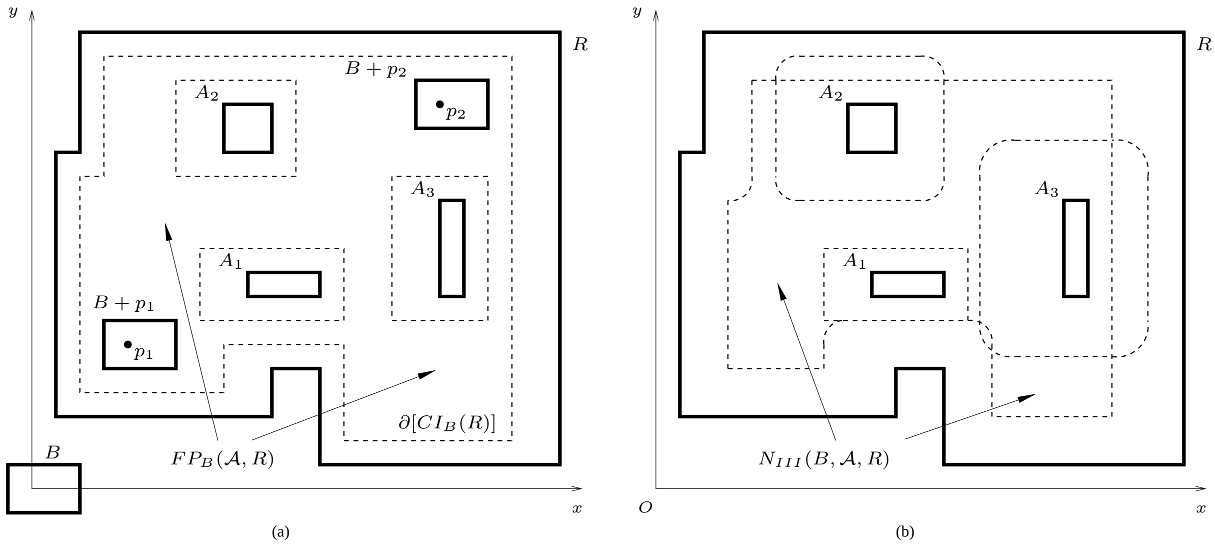


Figure 14: The translational Findspace problem in case where $\mathcal{A} = \{A_1, A_2, A_3\}$. (a) Here the region $FP_B(\mathcal{A}, R)$ is connected, and $p_{1,2}$ are the free positions of B. Dashed lines show $\partial[CO_B(\mathcal{A})]$ and $\partial[CI_B(R)]$, respectively. (b) Illustration for the Problem III. Here $N_{III}(B, \mathcal{A}, R)$ is a two-connected region and $\lambda_1 = 0$, $\lambda_2 = |\lambda_R|$, and $\lambda_3 = 1.5\cdot\lambda_2$, respectively. Dashed curves show $\partial[CO_B(\lambda_{\mathcal{A}}, \mathcal{A})]$ and $\partial[CI_B(\lambda_R, R)]$.

Since $[\gamma_1(B+p, A_i) > 0] = [\gamma_1(B+p, A_i) \le 0]^c$, we have $\bigwedge_{i=1}^n \big[\gamma_1(B+p, A_i) > 0\big] = \big\{\bigvee_{i=1}^n [\gamma_1(B+p, A_i) \le 0]\big\}^c$. Hence, the solving of the Findspace problem can be represented as follows:

$$\begin{aligned} N(B,\mathcal{A},R) &= H_1(0,K,B,R) \bigcap \left\{ \bigcap_{i=1}^n \Big[\Gamma_1(0,K,B,A_i)\Big]^c \right\} \\ &= H_1(0,K,B,R) \bigcap \left[\bigcup_{i=1}^n \Gamma_1(0,K,B,A_i) \right]^c \\ &= H_1(0,K,B,R) \bigcap \left[\Gamma_1\Big(0,K,B,\bigcup_{i=1}^n A_i\Big) \right]^c \\ &= H_1(0,K,B,R) \backslash \Gamma_1(0,K,B,\mathcal{A}). \end{aligned}$$

Clearly, $H_1(0, K, B, R) = CI_B(R)$ and $\Gamma_1(0, K, B, \mathcal{A}) = CO_B(\mathcal{A})$.

We next consider more general Findspace problems.

Problem III Find the corresponding region $N_{III}(B, \mathcal{A}, R)$ of the TGS

$$\nu_{III}(B+p,\mathcal{A},R) = \Big[\eta_1(B+p,R) \le \lambda_R \le 0\Big] \bigwedge \left\{ \bigwedge_{i=1}^n \Big[\gamma_1(B+p,A_i) > \lambda_i \ge 0\Big] \right\}.$$

Solving the Problem III. For given TGS $\nu_{III}(B+p, \mathcal{A}, R)$, the C-space obstacle depends on $\lambda_{\mathcal{A}} = \{\lambda_i\}_{i=1}^n$, corresponds to the TGS $\bigvee_{i=1}^n [\gamma_1(B+p, A_i) \le \lambda_i]$, and therefore it can be represented as $CO_B(\lambda_{\mathcal{A}}, \mathcal{A}) = \bigcup_{i=1}^n \Gamma_1(\lambda_i, K, B, A_i)$. The interior C-space corresponding to the TGS $[\eta_1(B+p,R) \le \lambda_R \le 0]$ is the region $CI_B(\lambda_R,R) = H_1(\lambda_R,K,B,R)$. (See Sections 4, 5, and Figure 14(b).) Thus, we get

$$N_{III}(B,\mathcal{A},R) = H_1(\lambda_R,K,B,R) \bigcap \Big\{ \bigcap_{i=1}^{n} \Big[\Gamma_1(\lambda_i,K,B,A_i)\Big]^c \Big\}$$

$$= H_1(\lambda_R,K,B,R) \backslash \Big[\bigcup_{i=1}^{n} \Gamma_1(\lambda_i,K,B,A_i)\Big].$$

Problem IV Find the solution $N_{IV}(B,\mathcal{A})$ of the TGS

$$\nu_{IV}(B+p,\mathcal{A}) = \Big[\gamma_{l_1}(B+p,\mathcal{A}) \le \lambda_1\Big] \bigwedge \Big[\gamma_{l_2}(B+p,\mathcal{A}) > \lambda_2\Big], \text{ where } l_{1(2)} = 1,2;\ \lambda_{1(2)} \ge 0.$$

Solving the Problem IV. The general solution is the region

$$N_{IV}(B,\mathcal{A}) = \Gamma_{l_1}(\lambda_1,K,B,\mathcal{A}) \bigcap \Big[\Gamma_{l_2}(\lambda_2,K,B,\mathcal{A})\Big]^c = \Gamma_{l_1}(\lambda_1,K,B,\mathcal{A}) \backslash \Gamma_{l_2}(\lambda_2,K,B,\mathcal{A}).$$

By observations of [42] and of Section 8, in case where $l_{1(2)} = 1$ and $\lambda_{1(2)} \ge 0$, we have

$$\nu_{IV}(B+p,\mathcal{A}) = \Big\{ \bigvee_{i=1}^{n} \Big[\gamma_1(B+p,A_i) \le \lambda_1\Big] \Big\} \bigwedge \Big\{ \bigwedge_{i=1}^{n} \Big[\gamma_1(B+p,A_i) > \lambda_2\Big] \Big\};$$

$$N_{IV}(B,\mathcal{A}) = \Big[\bigcup_{i=1}^{n} \Gamma_1(\lambda_1,K,B,A_i)\Big] \backslash \Big[\bigcup_{i=1}^{n} \Gamma_1(\lambda_2,K,B,A_i)\Big],$$

whereas, for $l_{1(2)} = 2$ and $\lambda_{1(2)} \ge \min_{1 \le i \le n}\{R_{A_i \oplus \check{B}}\}$, we get

$$\nu_{IV}(B+p,\mathcal{A}) = \Big\{ \bigwedge_{i=1}^{n} \Big[\gamma_2(B+p,A_i) \le \lambda_1\Big] \Big\} \bigwedge \Big\{ \bigvee_{i=1}^{n} \Big[\gamma_2(B+p,A_i) > \lambda_2\Big] \Big\};$$

$$N_{IV}(B,\mathcal{A}) = \Big[\bigcap_{i=1}^{n} \Gamma_2(\lambda_1,K,B,A_i)\Big] \backslash \Big[\bigcap_{i=1}^{n} \Gamma_2(\lambda_2,K,B,A_i)\Big].$$

See Figure 15. Note that $\nu_{IV}(B+p,\mathcal{A}) = 0$ and $N_{IV}(B,\mathcal{A}) = \emptyset$, for $\lambda_1 \le \lambda_2$.

Finally, we consider the Findspace problem concerning the *covering* of objects.

Problem V Find the region $N_V(B+p,\mathcal{A})$ of all the possible coverings of object $\mathcal{A}$ by object $B+p$, for given TGS

$$\nu_V(B+p,\mathcal{A}) = \Big\{ \bigvee_{i=1}^{n} \Big[\delta_2(B+p,A_i) \le \varepsilon_i\Big] \Big\} \bigwedge \Big\{ \bigwedge_{i=1}^{n} \Big[\lambda_i^- \le \delta_1(B+p,A_i) \le \lambda_i^+ \le 0\Big] \Big\}.$$

The solution is

$$N_V(B+p,\mathcal{A}) = \Big[\bigcup_{i=1}^{n} \Delta_2(\varepsilon_i,K,B,A_i)\Big] \bigcap \Big\{ \bigcap_{i=1}^{n} \Big[\Delta_1(\lambda_i^+,K,B,A_i) \backslash \mathbf{i}\Delta_1(\lambda_i^-,K,B,A_i)\Big] \Big\}.$$

For more simple constraint $\nu_V(B+p,\mathcal{A}) = \bigwedge_{i=1}^{n}[\delta_1(B+p,A_i) \le \lambda_i \le 0]$, we get $N_V(B,\mathcal{A}) = \bigcap_{i=1}^{n} \Delta_1(\lambda_i,K,B,A_i)$; see Figure 16.

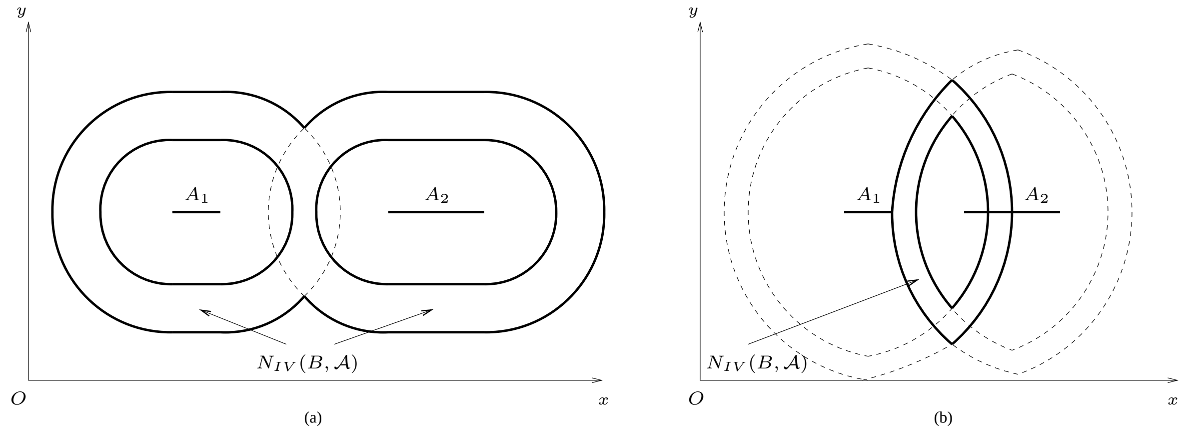


Figure 15: Illustration for the Problem IV. Here $\mathcal{A} = \{A_1, A_2\}$, both A_1 and A_2 are line segments, $B = \{O\}$, $\lambda_1 > \lambda_2$, and the region $N_{IV}(B, \mathcal{A})$ is connected. Dashed curves show pieces of $\partial\Gamma_{l_1(l_2)}(\lambda_{1(2)}, B, A_{1(2)})$. Here (a) $l_{1(2)} = 1$, and (b) $l_{1(2)} = 2$.

9.2 Placement of geometric objects

An approches commonly used for solving the placement problems are the *sequential-single* method, suggested in [53] and [57], and the *multiple* placement of several objects, suggested in [2], [9], and [33].

The sequential-single placement consists in the sequential placement of the objects of $\mathcal{A}$ with respect to the *container* A_0 in a fixed order, say $A_1, A_2, \ldots, A_n$, according to the special objective function. For instance, the valid position p_i of A_i must be a point with extremal (or specified) values of coordinates. (In case of planar problems $p_i = (x_i, y_i)$ can have, for example, a minimal, maximal, or specified x_i and/or y_i.)

The multiple (or simultaneous) placement, as follows from its name, is independent on the order of placement of the objects of $\mathcal{A}$ relative to A_0, and provides the placement of each object of $\mathcal{A}$, say A_j, taking into account the possibility of the placement of all another objects $\{A_i\}$, where $1 \leq i \leq n$, and $i \neq j$. The goal is to find the set P_{0j} of all the feasible positions of A_j, for $j = 1, \ldots, n$, with respect to A_0.

The generalized sequential-single and multiple placement problems (i.e., the problems with more complex constraints on the relative position of objects than in [33] and [57]) and their solutions have been studied in [42].

9.3 Application to the other types of geometric situations

The work in [41] has studied the following types of geometric situations:

The *translational* geometric situation *in direction* u describes constraints on translational distances *in a given direction* between geometric objects. The minimum and maximum distances $\gamma_{1,2}(B, A, u)$ taking into account the *outer* position of B relative to A in direction u have been introduced in [22] and [37]. The minimum and maximum distances $\eta_{1,2}(B, A, u)$ taking into account the *inner* position of B relative to A have been proposed in [39] and [41]. The parametric families of objects corresponding to the distances in a given direction are obtained

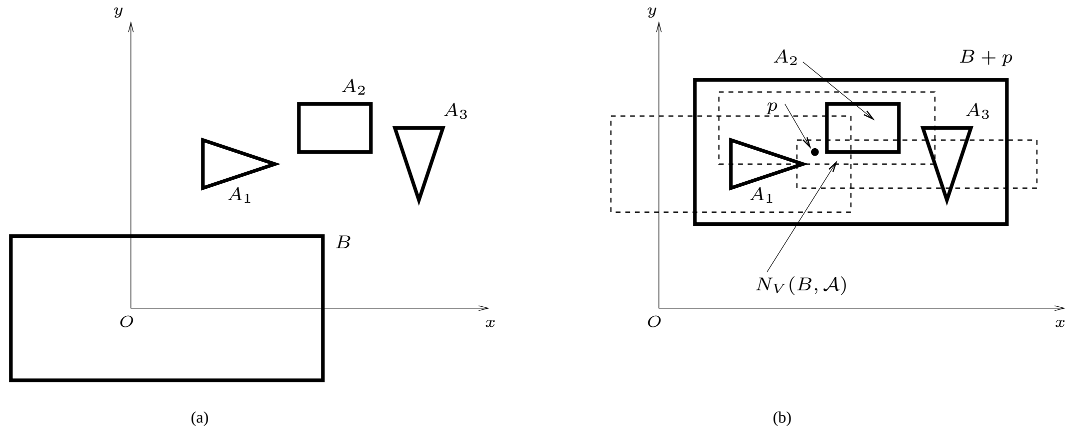


Figure 16: Illustration for the Problem V. (a) An objects $\mathcal{A} = \{A_1, A_2, A_3\}$ and B. (b) Here the region $N_V(B, \mathcal{A})$ is siply connected, $\mathcal{A} \subset (B+p)$, for $p \in N_V(B, \mathcal{A})$; $\lambda_1 = 0$, $\lambda_2 < 0$, and $\lambda_3 = \lambda_2$, respectively. Dashed lines show $\partial \Delta_1(\lambda_{1-3}, B, A)$.

by using the *partial* vector operations on objects, which are generalizations of the Minkowski operations. See [41] and [45] for more detailes.

The *rotational* geometric situation describes constraints on minimum and maximum *rotational* distances between geometric objects. Denote by A^* the image of A in polar coordinates, i.e., $A^* = \{(r,\theta) \mid (r\cos\theta, r\sin\theta) \in A\}$. Then a copy A^ϕ of A rotated by an angle ϕ around the origin O corresponds to a copy $A^* + \{(0,\phi)\}$ of A^* translated by a point $(0,\phi)$ in polar coordinates. Therefore one can use the *partial* vector operations in polar coordinates for modeling the relative position of geometric objects under rotations. The rotational distances and the translational distances in a given direction have used in [41] to formalize the constraints on the relative position of links in problems of modeling of mechanism's motion.

The *dynamic* geometric situation is defined, for moving objects, by representing an objects as a four-dimensional sets in the space-time G^4; see [6], [39], and [41]. (Note that the distances between objects "in the space" and "in the time" are incomparable [64].)

Let $A^* = \bigcup_{t\in[0,1]}[A(t)]$, $B^* = \bigcup_{t\in[0,1]}[B(t)]$ be the image of A, B in G^4. We denote by $\oplus^t$ the partial addition "by the time", and by $\check{B}^*$ the reflection of B^* with respect to the origin O in R^3. Let next λK^* be the cylinder in G^4 with a basis λK. Then the parametric family of objects

$$\Gamma_1(\lambda, K, B^*, A^*) = \bigcup_{t\in[0,1]} [A(t) \oplus \check{B}(t) \oplus \lambda K] = A^* \overset{t}{\oplus} \check{B}^* \overset{t}{\oplus} \lambda K^*$$

corresponds to the distance $\gamma_1(B^*, A^*) = \inf_{t\in[0,1]}\{\gamma_1[B(t), A(t)]\}$ between A^* and B^*.

It is clear that the suggested technique for solving the translational spatial planning problems can also be applied to geometric problems with the considered types of constaints on the relative position of objects.

10 Computational issues

From the observations of Sections 8 and 9 it follows that the C-space map of spatial planning problem, is the region obtained by standard and/or regularized Boolean operations, and by

Minkowski operations on regular and/or non-regular objects. In general, the C-space map is a *non-regular* geometric object of various topology with *non-manifold* boundary; see Remark 5 of Section 8. (Recall that the non-regular object may have external dangling faces/edges and/or isolated points, and/or internal entities such as cracks and/or isolated points [43].) Therefore for implementation of spatial planning we need the methods for representation and manipulation of such an objects. In this section we briefly consider the computer representations of geometric objects that are suitable for solving the spatial planning problems, and the strategies for computing the C-space maps.

Representaions of geometric objects. The two representation schemes that are most widely used in solid modeling and computer graphics are boundary representation (BRep) and constructive solid geometry (CSG) [43]. Let A be a point set of R^n ($n = 2, 3$). CSG(A) is a Boolean composition of algebraic halfspaces using regularized set operations. BRep(A) is a collection of closed faces/edges. The problems of CSG to BRep conversion and of BRep to CSG conversion have been studied in [43], [50], and [51].

The third type of representation scheme suitable for our purpose is the linear ray representation (LRRep) [31], [39], denoted by LRR(A). (In [39] it is called the linear raster representation.) LRR(A) is an approximation of an object A by a set of parallel segments belonging to a grid L of parallel lines, i.e., LRR(A)= $A \cap L$. Conversions between BRep and LRRep, and between CSG and LRRep have been detally studied in [32].

The constructive non-regularized geometry (CNRG) methodology for representation and manipulation of non-homogeneous (i.e., made of several materials with different properties), non-closed point sets with internal structures and incomplete boundaries have been suggested in [48]. The work in [18] has proposed an approach for representation of geometric objects with non-manifold boundary.

The boundary representation of non-regular geometric objects with non-manifold boundary using the techniques of [18] and [48] have been studied in [42]. It takes into account both the geometry and the topology of objects. The work in [42] have also considered the topological operations (complement, interior, closure, and regularization) on a single non-regular object.

Boolean operations. Algorithms and implementation for computing the regularized set operations on polyhedral objects have been proposed in [44]. Recall that the standard union $A \cup B$ of two r-sets A and B always results in an r-set, whereas the standard intersection $A \cap B$ and the standard set difference $A \backslash B$ need not be regular: the set $A \cap B$ may have dangling edges, e.g., in case where A contacts with B along the portion of its boundary [43]; the set $A \backslash B$ may be partially open, e.g., in case where $\mathbf{i}A \cap \mathbf{i}B \neq \emptyset$ [36], [44]. Algorithms for computing the set operations on non-manifold boundary representation objects have been proposed in [18] and [46]. We assume below some familiarity with theory and algorithms of [18] and [44].

As mentioned in [44], the algorithms will work with curved objects, and they are insensitive to whether a solid's boundary is or is not a two-manifold, and is or is not connected. Hence, the algorithms of [44] can be modified to compute the standard Boolean operations on non-regular objects of various topology with non-manifold boundary.

Let $S = A \otimes^* B$, where $\otimes$ denotes one of the standard set operations. The main utilities used in algorithms of [44] to compute the boundary of S are the set membership classification (SMC)

[59], and the combining classifications, defined by means of the regularized set operations. See [44] and [59] for details.

In [42] the algorithms of [44] have been modified for computing the standard Boolean set operations on non-regular (possibly unbounded) geometric objects of various topology, for BReps. To define and to combine the classifications the work in [42] have used the standard, but not a regularized set operations.

Minkowski operations. Many various algorithms to compute the Minkowski operations have been proposed. Detailed surveys of previous work on computing the Minkowski operations can be found in [13], [20], [25], [26], [36], [60], and [63]. Algorithms for computing the Minkowski sums and the Minkowski differences in two and three dimensions are given, e.g., in [2], [3], [14], [15], [17], [25] – [29], [33], [37], and [60], for BReps, and in [31] and [39], for LRReps. Note that the referenced algorithms perform computing the Minkowski operations on regular objects. They can generate the manifold boundaries and are not applicable to the cases where the boundary of the resulting object is non-manifold; see [60] for details.

In [12] and [20] have been presented an algorithms for *robust* and *efficient* construction of planar Minkowski sums for polygons using exact rational arithmetic. In contrast with most existing techniques the algorithms of [12], [20] directly handle the degenerate configurations, arising in the boundary of the Minkowski sum, such as internal isolated points and/or coinciding edges. In other words, these algorithms compute the outer envelope of A and B, i.e., the boundary of the open set $\mathbf{i}A \oplus \mathbf{i}B$ (see subsection 2.1). The recent works in [61] and [62] have presented an algorithms for exact and efficient construction of Minkowski sums of polygons, and for exact and approximate construction of offset polygons, respectively, that handle the degenerate configurations also. Hence, the algorithms of [12], [20], [61], and [62] allow to construct the parametric families of polygonal objects used for computing the C-space maps.

The algorithms of [12] and [20] are based on convex decomposition of polygons. However, as mentioned in [3], not all curved objects permit convex decomposition, e.g., an object with an inward concave edge. Therefore to handle the curved objects more suitable are the methods that deal with the geometric objects directly.

In [42] the algorithms of [3] have been modified for computing the Minkowski operations on non-regular objects of various topology, for BReps, using the techniques of [18] and [48].

Distances between geometric objects. Many algorithms for computing the distances between geometric objects have been developed (see [27]). Detailed surveys of previous work on computing the MTD between regular objects can be found in [10], [13], [16], [21], [27], and [35]. Algorithms for computing the distances concerning the outer relative position of objects in two and three dimensions (see subsection 2.2) are given, e.g., in [7], [10], [13] – [17], [35], [37], and [52], for BReps, and in [39] and [41], for LRReps. The algorithms of [12], [20], and [61] for robust construction of planar Minkowski sums can be used for computing the MTD between non-regular polygonal objects.

Algorithms for computing the distances concerning the containment of objects in two and three dimensions (see subsection 5.1) are given in [39] and [41], for LRReps. Note that for this goal can also be used algorithms for computing the Minkowski difference, for BReps; see, e.g., [14] and [15]. Algorithms for computing the distances concerning the covering of objects (see subsection 5.2) and algorithms for computing the distances concerning the containment of objects are similar.

Algorithms for computing the translational distances in a given direction in two and three dimensions are given in [10], [13], [22], and [52], for BReps, and in [39] and [41], for LR-Reps, respectively. The algorithms of [39] and [41] are based on the partial vector operations (see subsection 9.1). Algorithms for computing the rotational distances and the partial vector operations in polar coordinates are given in [41].

Thus, using the algorithms for computing the various types of distances between geometric objects provide solving the generalized distance query problem, as defined in subsection 1.3.

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