Factorization of Rational Curves in the Study Quadric and Revolute Linkages

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Given a generic rational curve C in the group of Euclidean displacements we construct a linkage such that the constrained motion of one of the links is exactly C. Our construction is based on the factorization of polynomials over dual quaternions. Low degree examples include the Bennett mechanisms and contain new types of overconstrained 6R-chains as sub-mechanisms.

Keywords: Dual quaternions, rational motion, factorization, Bennett mechanism, over-constrained mechanism, 6R-chain.

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1 Introduction

The research on this paper started with an attempt to understand the geometry of Bennett linkages [1–6] from the point of view of dual quaternions. The group of Euclidean displacements can be embedded as an open subset of the Study quadric in the projectivization of the dual quaternions regarded as a real vector space of dimension eight. Rotation subgroups and and their composition then get an algebraic meaning (see Hao [7], Selig [8, Chapter 9 and 10]). This was exploited in [4] to devise an algorithm for the synthesis of a Bennett linkage to three pre-assigned poses. The key observation there was that the coupler curve is the intersection of a unique 2-plane with the Study quadric. Here, we translate the synthesis problem entirely into the language of dual quaternions and we show that the problem is equivalent to the factorization of a left quadratic polynomial into two linear ones.

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Factorizations of left polynomials over the quaternions have been studied by Niven in [9], who was interested in the number of such factorizations, and Gordon and Motzkin in [10], who proved that this number is either infinite or at most equal to the factorial of the degree of the polynomial. ([10] studies more generally polynomials over central simple algebras over commutative fields.) The more recent paper [11] by Huang and So gives an explicit solution formula for quadratic polynomials. It is not difficult to extend these results to dual quaternions.

Once the relation between the closure conditions in linkages and the factorizations of left polynomials over dual quaternions became clear, it also became clear that this relation holds for arbitrary linkages with rotational joints. Given a generic rational curve C of degree n on the Study quadric with parametrization P(t), we construct n! different factorizations $P(t) = (t - h_1) \cdots (t - h_n)$ with rotation quaternions h_1, \ldots, h_n (or translation quaternions in limiting cases). Each factorization corresponds to the movement of an open nR-chain that guides the end-effector along C (Section 3). Combining all open chains yields an overconstrained mechanism whose combinatorial structure is investigated in Section 4. In Section 5, we discuss the extension to linkages with prismatic joints.

The case n=2 is just the Bennett case. Our results agree with the recent findings of [12] and naturally include the limit case of Bennett linkages (RPRP linkages). For n=3 we obtain linkages that contain new examples of overconstrained 6R-chains. We present them in more detail in Section 6.

A preliminary short version of this article is [13]. The present article is more complete. We give strict proofs for all new results, discuss the extension to prismatic joints, present additional examples and illustrate our results by figures. The Maple source for some of the examples and additional animations can be found on the accompanying web-site http://geometrie.uibk.ac.at/schroecker/qf/.

2 Preliminaries

In this section, we recall the well-known and classical description of the group of Euclidean displacement by dual quaternions. The presentation is already adapted to our needs in later sections. More general references are [7, 8].

We denote by SE_3 the group of direct Euclidean displacements, i.e., the group of maps from \mathbb{R}^3 to itself that preserves distances and orientation. It is well-known that SE_3 is a semidirect product of the translation subgroup T and the special orthogonal group SO_3 , which may be identified with the stabilizer of a single point.

We denote by $\mathbb{D} := \mathbb{R} + \epsilon \mathbb{R}$ the ring of dual numbers, with multiplication defined by $\epsilon^2 = 0$. The algebra \mathbb{H} is the non-commutative algebra of quaternions, and $\mathbb{D}\mathbb{H}$ is the algebra of quaternions with coefficients in \mathbb{D} . Every dual quaternion has a primal and a dual part (both quaternions in \mathbb{H}), a scalar part in \mathbb{D} and a vectorial part in \mathbb{D}^3 . The conjugated dual quaternion \overline{h} of h is obtained by multiplying the vectorial part of h by -1. The dual numbers $h\overline{h}$ and $h + \overline{h}$ are called the *norm* and *trace* of h, respectively.

By projectivizing \mathbb{DH} as a real 8-dimensional vectorspace, we obtain \mathbb{P}^7 . The condition that the norm of h is strictly real, i.e. its dual part is zero, is a homogeneous quadratic

equation. Its zero set, denoted by S, is called the Study quadric. The linear 3-space of all dual quaternions with zero primal part is denoted by E. It is contained in the Study quadric. The complement S - E can be identified with SE₃. The primal part describes SO₃. Translations correspond to dual quaternions with primal part ± 1 and strictly vectorial dual part. More precisely, the group isomorphism is given by sending $h = p + \epsilon q$ to the map

 $v \in \mathbb{R}^3 \mapsto \frac{pv\overline{p} + q\overline{p}}{p\overline{p}} \in \mathbb{R}^3.$

The image of this map is strictly vectorial, the map is in SE₃, and the above formula indeed provides a group homomorphism. Its bijectivity follows from the fact that both groups are connected and of the same dimension.

A nonzero dual quaternion $h = p + \epsilon q$ represents a rotation if and only if its norm and trace are strictly real $(h\overline{h}, h + \overline{h} \in \mathbb{R})$ and its primal vectorial part is nonzero $(p \notin \mathbb{R})$. It represents a translation if and only if its norm and trace are strictly real and its primal vectorial part is zero $(p \in \mathbb{R})$. The 1-parameter rotation subgroups with fixed axis and the 1-parameter translation subgroups with fixed direction can be characterized as lines on S through the identity element 1. Translations are characterized as those lines that meet the exceptional 3-plane E.

3 Rational motions and open linkages

In this section we study rational curves in the Study quadric and construct a linkage with rotational joints such that the last link moves along the prescribed curve. The main technique is the factorization of polynomials over the dual quaternions.

3.1 Polynomial factorization over dual quaternions

Let $C \subset S$ be a rational curve of degree n > 0. Then there exists a parametrization of C by a polynomial $(a_nt^n + a_{n-1}t^{n-1} + a_{n-2}t^{n-2} + \cdots + a_0)_{t \in \mathbb{P}^1}$, where $a_0, \ldots, a_n \in \mathbb{DH}$ and \mathbb{P}^1 denotes the real projective line. It is no loss of generality to assume that this polynomial is monic $(a_n = 1)$. This can always be achieved by an appropriate choice of coordinates.

Conversely, let $\mathbb{DH}[t]$ be the set of left polynomials with coefficients over \mathbb{DH} . This set can be given a ring structure by the convention that t commutes with the coefficients. Let $P \in \mathbb{DH}[t]$ be a polynomial of degree n > 0. We call the map $f_P: \mathbb{P}^1 \to \mathbb{P}^7$, $t \mapsto P(t)$ the map associated to P. The image is a rational curve of degree at most n.

If $Q \in \mathbb{R}[t]$, $Q \neq 0$, then the maps f_P and f_{PQ} are equal. Conversely, if P has a factor in $\mathbb{R}[t]$ of positive degree, we can divide by it without changing the associated map. Note that, in general, the set of right factors is different from the set of left factors. However, a polynomial in $\mathbb{R}[t]$ or in $\mathbb{D}[t]$ is a left factor if and only if it is a right factor, because it is in the center of $\mathbb{DH}[t]$.

For n > 0, an open linkage with n rotational joints can be described algebraically as follows. Let h_1, \ldots, h_n be rotations; for each i, the group parametrized by $(t - h_i)_{t \in \mathbb{P}^1}$ – the parameter t determines the rotation angle – is the group of the (i + 1)-th link relative

to the i-th link. If we choose the same parameter for the n rotations, then the position of the last link with respect to the first link is given by a product

$$P = (t - h_1)(t - h_2) \cdots (t - h_n) \quad \text{with} \quad t \in \mathbb{P}^1.$$

P is a monic polynomial of degree n. Because it describes a rigid motion, it satisfies the Study condition $P\overline{P} \in \mathbb{R}[t]$. In this case we call P a motion polynomial. We ask the converse question: Given a monic motion polynomial P of degree n, is it possible to construct a factorization of type (1)? We will see that, in general, the answer is positive (Theorem 3, below).

For the time being we restrict ourselves to factorizations with rotation quaternions only. Later, in Section 5, we will show how to incorporate translation quaternions as well.

Lemma 1. If h is a rotation quaternion then $M := (t - h)(t - \overline{h})$ is in $\mathbb{R}[t]$ and has no real roots.

Proof. The first claim follows from the expansion $M = t^2 - t(h + \overline{h}) + h\overline{h}$ and the observations that $h + \overline{h} \in \mathbb{R}$ (because the scalar dual part of h is zero) and $h\overline{h} \in \mathbb{R}$ (because h lies on the Study quadric).

The discriminant of M equals $\Delta = (h + \overline{h})^2 - 4h\overline{h} = (h - \overline{h})^2$. Since h is a rotation quaternion, Δ is negative and M has no real roots. (The case $\Delta = 0$ characterizes translations.)

Proposition 2. Let $P \in \mathbb{DH}[t]$ be a motion polynomial of degree n > 0 without strictly real factors. If there is a factorization $P(t) = (t - h_1) \cdots (t - h_n)$ with rotation quaternions $h_1, \ldots, h_n \in \mathbb{DH}$, the polynomial $P\overline{P} \in \mathbb{R}[t]$ has no real zeroes.

Proof. Assume $P=(t-h_1)\cdots(t-h_n)$ and let $M_i:=(t-h_i)(t-\overline{h_i})$ for $i=1,\ldots,n$. By Lemma 1, M_i is in $\mathbb{R}[t]$ and has no real roots. The same is true for $P\overline{P}=(t-h_1)\cdots(t-h_n)(t-\overline{h_n})\cdots(t-\overline{h_1})=M_1\cdots M_n$.

We call a motion polynomial $P \in \mathbb{DH}[t]$ of degree n > 0 generic if $P\overline{P}$ has n distinct quadratic, irreducible factors. This implies that primal(P) has no strictly real factors, because any such factor would appear with multiplicity ≥ 2 in the factorization of $P\overline{P}$.

Theorem 3. Let $P \in \mathbb{DH}[t]$ be a generic motion polynomial of degree n > 0. Then there exists a factorization $P(t) = (t - h_1) \cdots (t - h_n)$ with $h_i \in \mathbb{DH}$ representing rotations.

We divide the proof of Theorem 3 into several lemmas. A central tool (also for the computation of the factorization) is polynomial division in $\mathbb{DH}[t]$.

Lemma 4 (polynomial division). Let $P_1, P_2 \in \mathbb{DH}[t]$ and assume P_2 is monic. Then there exists a unique representation $P_1 = QP_2 + R$ with $Q, R \in \mathbb{DH}[t]$ and $\deg(R) < \deg(P_2)$. Moreover, if $h \in \mathbb{DH}$ such that $P_2(h) = 0$, then $P_1(h) = R(h)$.

Proof. The proof is a generalization of polynomial division to the non-commutative case. Let $n := \deg(P_1)$ and $m := \deg(P_2)$. Existence of the representation is trivial if n < m $(Q = 0, R = P_1)$. Assume inductively that polynomial division is possible for

all polynomials of degree less than n. We can write $P_1 = at^n + P_3$ with $\deg(P_3) < n$, and $at^n = at^{n-m}P_2 + P_4$ with $\deg(P_4) < n$. By the induction hypothesis, we have $P_3 = Q_3P_2 + R_3$ and $P_4 = Q_4P_2 + R_4$ with $\deg(R_3), \deg(R_4) < m$. Combining, we obtain $P_1 = (at^{n-m} + Q_4 + Q_3)P_2 + R_3 + R_4$. This shows existence.

Now, assume that $P_1 = Q_1P_2 + R_1 = Q_2P_2 + R_2$. Then $(Q_1 - Q_2)P_2 = R_1 - R_2$. If $Q_1 \neq Q_2$, the polynomial on the left has degree greater or equal m while the polynomial on the right has degree less than m. This is impossible, so that $Q_1 = Q_2$ and $R_1 = R_2$. This shows uniqueness.

For the last statement, we have to show that $P_2(h) = 0$ implies $(QP_2)(h) = 0$. This is not trivial, because h does not commute with the coefficients of P_2 . But the statement is linear in Q, hence it is enough to prove it for monomials: If $P_2(h) = 0$ and $Q = at^r$, then $(QP_2)(h) = aP_2(h)h^r = 0$.

Lemma 5. Let $P \in \mathbb{DH}[t]$ and $h \in \mathbb{DH}$. Then (t - h) is a right factor of P if and only if P(h) = 0.

Proof. By Lemma 4, there is a unique representation P = Q(t - h) + R, and R is a constant equal to P(h).

Lemma 6. Let $P \in \mathbb{DH}[t]$ be a motion polynomial. Let $M \in \mathbb{R}[t]$ be a monic polynomial of degree two that divides $P\overline{P}$ but not primal(P). Then there exists a unique $h \in \mathbb{DH}$ such that P(h) = M(h) = 0.

Proof. By Lemma 4, we may write P = QM + R with $Q, R \in \mathbb{DH}[t]$, $\deg(R) < 2$. Since M does not divide primal(P), we have primal $(R) \neq 0$. On the other hand,

$$P\overline{P} = (QM + R)(M\overline{Q} + \overline{R}) = (Q\overline{Q}M + Q\overline{R} + R\overline{Q})M + R\overline{R}.$$

Because M divides $P\overline{P}$, we conclude $R\overline{R} = cM$ for some $c \in \mathbb{D}$, $c \neq 0$. Assume that the primal part of c is zero, that is, $R\overline{R} = \epsilon kM$ with $k \in \mathbb{H}$. This implies $\operatorname{primal}(R) = 0$ and contradicts our assumption. Hence $\operatorname{primal}(c) \neq 0$. The leading coefficient r_1 of R cannot have zero primal part because otherwise we get the contradiction $r_1\overline{r}_1 = 0 = c$. Hence, r_1 is invertible in \mathbb{DH} and, because of $r_1\overline{r}_1 = c$, so is c. The polynomial $R = r_1t + r_0$ is linear with invertible leading coefficient. Hence it has a unique zero $h = -r_1^{-1}r_0 \in \mathbb{DH}$. From R(h) = 0, we obtain cM(h) = 0 and, because c is invertible, M(h) = P(h) = 0. This shows existence.

In order to show uniqueness, assume there exists $h' \in \mathbb{DH}$ with P(h') = M(h') = 0. This implies R(h') = 0. But then h' = h because the zero of R is unique.

Proof of Theorem 3: We proceed by induction on n. For n=0, the statement is trivial. Assume $n \geq 1$. Since the primal part of P has no strictly real factors, P itself has no strictly real factors. Consequently, $P\overline{P}$ has no linear factors.

Let M be one of the irreducible quadratic factors of $P\overline{P}$. By Lemma 6, there is a unique h such that M(h) = P(h) = 0. By Lemma 5, there exists $Q \in \mathbb{DH}[t]$ such that P = Q(t-h). Obviously, Q is monic of degree n-1. Moreover, we have $P\overline{P} = Q(t-h)(t-\overline{h})\overline{Q} = Q\overline{Q}M$ so that $Q\overline{Q}$ is in $\mathbb{R}[t]$. Furthermore, Q cannot have a strictly real

factor: This factor would also be a left factor and hence divide P. For similar reasons, the primal part of Q cannot have strictly real factor (it would divide P). By induction hypothesis, we obtain $Q = (t - h_1) \cdots (t - h_{n-1})$ and so $P = (t - h_1) \cdots (t - h_{n-1})(t - h)$. Because of $P\overline{P} \in \mathbb{R}[t]$, h must be a rotation or translation quaternion. By genericity of P, it is actually a rotation quaternion.

Remark 7. Theorem 3 is almost a converse to Proposition 2. But there are polynomials P without real factors such that $P\overline{P} \in \mathbb{R}[t]$ where the proposition and the theorem do not say anything. In this case, there exists an irreducible polynomial $R \in \mathbb{R}[t]$ dividing the primal part of P but not the dual part. Since R then also divides the primal part of \overline{P} and $P\overline{P} \in \mathbb{R}[t]$, the factor R appears with multiplicity two in the factorization of $P\overline{P}$. For instance, if $P = t^2 + 1 + \epsilon \mathbf{i}$, it can be shown that P is not the product of two linear rotation polynomials. (Here, we assume that $(1, \mathbf{i}, \mathbf{j}, \mathbf{k})$ is the standard basis of the quaternion algebra \mathbb{H} .) On the other hand, $P = t^2 + 1 + \epsilon \mathbf{i} = (t - \mathbf{k})(t - \mathbf{k} + \epsilon \mathbf{j})$ is a product of two rotation polynomials. A systematic analysis would be good, but it is probably more difficult. At this place, we only observe that $P = t^2 + 1 + \epsilon \mathbf{i}$ is a quadratic parametrization of a straight line.

3.2 Computing the factorization

Our proof of Theorem 3 is constructive and, with exception of the factorization of $P\overline{P}$, can be implemented in rational arithmetic. We describe this in more detail. Our considerations will lead us to two new insights:

- The factorization depends on an ordering of the n quadratic factors of the polynomial $P\overline{P}$. Hence, there exist n! different factorizations and Theorem 3 actually admits a stronger version (Theorem 8, below).
- If one factorization of P is known, the remaining factorizations can be computed in rational arithmetic. In particular, it is possible to construct completely rational examples by starting with the polynomial $P = (t h_1) \cdots (t h_n)$ where h_1, \ldots, h_n are rotation quaternions with rational coefficients.

Given is a motion polynomial $P \in \mathbb{DH}[t]$. We want to compute rotation quaternions h_1, \ldots, h_n such that $P = (t - h_1) \cdots (t - h_n)$. The pseudo-code for this calculation is given in Algorithm 1. It returns an n-tuple $H = (h_1, \ldots, h_n)$. Here are some remarks on the actual calculations:

- In Line 4, the choice of $M_i \in M$ is arbitrary. Thus, Algorithm 1 is not deterministic.
- The common root h_i of M_i and P in Line 5 is found by computing the remainder $R = r_1 t + r_0$ of the polynomial division of P by M_i and setting $h_i = -r_1^{-1} r_0$. This is justified in the proof of Lemma 6.
- The proof of Theorem 3 shows that the polynomial division of P by $(t h_i)$ is possible without remainder. Moreover, the quotient satisfies all requirements of Theorem 3 so that it is suitable as input for yet another repeat-loop.

Algorithm 1 Compute factorization $P = (t - h_1) \dots (t - h_n)$

Require: generic motion polynomial P

- 1: $M \leftarrow \{M_1, \dots, M_n\}$ where $P\overline{P} = M_1 \cdots M_n$.
- 2: Let H denote the empty tuple.
- 3: repeat
- 4: Choose $M_i \in M$ and set $M \leftarrow M \setminus \{M_i\}$.
- 5: Compute h_i such that $M_i(h_i) = P(h_i) = 0$.
- 6: Append h_i to H.
- 7: $P \leftarrow P/(t h_i)$
- 8: **until** $\deg P = 0$
- 9: **return** $H = (h_1, \ldots, h_n)$

Maple source code for Algorithm 1 can be found on the web-site http://geometrie.uibk.ac.at/schroecker/qf/. The non-deterministic nature of Algorithm 1 implies existence of different factorizations into products of rotation quaternions.

Theorem 8. Let $P \in \mathbb{DH}[t]$ be a generic motion polynomial of degree n > 0. Then there is a one-to-one correspondence between factorizations $P(t) = (t - h_1) \cdots (t - h_n)$ into linear polynomials over \mathbb{DH} and permutations of the n distinct quadratic irreducible factors M_1, \ldots, M_n of $P\overline{P}$. In any of these factorizations we have $M_i(h_i) = 0$ for $i = 1, \ldots, n$.

Proof. Our proof of Theorem 3 can be translated into a construction of a factorization of P into linear factors over \mathbb{DH} . The only non-deterministic step is the choice of a quadratic factor of $P'\overline{P'}$, where P' is the left factor from which the next right linear factor is going to be constructed. The construction is also complete in the sense that every factorization can be obtained by this non-deterministic algorithm.

Example 1. We give an example that illustrates some of the results we obtained so far. Consider the quadratic polynomial

$$P = t^2 - t(1 + (\epsilon - 1)\mathbf{i} + (1 - \epsilon)\mathbf{j} + 2(1 + \epsilon)\mathbf{k}) - 1 - 2\epsilon + \mathbf{i} - \epsilon\mathbf{j} + (2 - \epsilon)\mathbf{k}.$$

We can write $P\overline{P} = M_1 M_2$ where

$$M_1 = t^2 + 2$$
 and $M_2 = t^2 - 2t + 3$.

In order to find the common root h_2 of P and M_1 , we compute the remainder of the polynomial division of P by M_1 . It is $R = r_1t + r_0$ where

$$r_1 = -1 + (1 - \epsilon)\mathbf{i} + (\epsilon - 1)\mathbf{j} - 2(1 + \epsilon)\mathbf{k}, \quad r_0 = -3 - 2\epsilon + \mathbf{i} - \epsilon\mathbf{j} + (2 - \epsilon)\mathbf{k}.$$

Now

$$h_2 = -r_1^{-1}r_0 = \left(\frac{-4}{7} + \frac{30}{49}\epsilon\right)\mathbf{i} - \left(\frac{1}{7} + \frac{3}{49}\epsilon\right)\mathbf{j} + \left(\frac{9}{7} + \frac{13}{49}\epsilon\right)\mathbf{k}$$

and, by polynomial division, we find $P = (t - h_1)(t - h_2)$ with

$$h_1 = 1 + \left(\frac{-3}{7} + \frac{19}{49}\epsilon\right)\mathbf{i} + \left(\frac{8}{7} + \frac{-46}{49}\epsilon\right)\mathbf{j} + \left(\frac{5}{7} + \frac{85}{49}\epsilon\right)\mathbf{k}.$$

Note that $P(h_2) = M_1(h_2) = 0$.

Using M_2 instead of M_1 , we can compute a second factorization $P = (t - h'_1)(t - h'_2)$ where

$$h'_1 = (1 - \epsilon)\mathbf{j} + (1 + \epsilon)\mathbf{k},$$

$$h'_2 = 1 - (1 - \epsilon)\mathbf{i} + (1 + \epsilon)\mathbf{k}.$$

Here, $P(h_2') = M_2(h_2') = 0$.

We already showed how to compute the rotation quaternion h_i from the factor M_i . But it is also possible to compute M_i from h_i . Clearly, M_i is the unique monic polynomial of degree two such that $M_i(h_i) = 0$ (the minimal polynomial of h_i). It is given by $M_i = t^2 - (h_i + \overline{h_i})t + h_i\overline{h_i}$. This observation is important, because it allows us to construct completely rational examples. Setting $P = (t - h_1) \cdots (t - h_n)$ with $h_1, \ldots, h_n \in \mathbb{DH}$ we can directly compute the minimal polynomials M_1, \ldots, M_n and also the factorization in Line 1 of Algorithm 1. The polynomials in our examples were actually obtained in this way.

3.3 Kinematic interpretation

Now we are going to translate Theorem 3 into the language of kinematics. A precise formulation takes into account the possibility of neighboring factors $t - h_i$ and $t - h_{i+1}$ that describe rotations about the same axis. In this case, we call h_i and h_{i+1} compatible. This is the case if and only if $h_i h_{i+1} = h_{i+1} h_i$.

Let $h \in \mathbb{DH}$ be a dual quaternion representing a rotation. The parametrization $(t-h)_{t\in\mathbb{P}^1}$ of the rotation group defined by h is called a linear parametrization. More generally, let $R_1, R_2 \in \mathbb{R}[t]$ such that R_1 is monic, $\deg(R_1) = n$, $\deg(R_2) < n$, without common factor. Then the parametrization $(R_1(t) - h_1R_2(t))_{t\in\mathbb{P}^1}$ is called a rational parametrization of degree n. Higher degree parametrizations of rotation groups may arise as the product of linear parametrizations, if the two axes coincide, that is, if the rotation quaternions are compatible.

A rational curve $C \subset S$ of degree n in the Study quadric admits a parametrization by a motion polynomial P of degree n. We say that C is a generic rational curve of degree n in the Study quadric if P is generic (its norm polynomial has n distinct irreducible quadratic factors). It is straightforward to show that this notion of genericity of C is well-defined, i.e., it does not depend on the choice of the motion polynomial P.

Corollary 9. Let $C \subset S$ be a generic rational curve of degree n in the Study quadric, passing through 1. Then C can be obtained as movement of the last link of an open kR-linkage, with $k \leq n$. The rotations in the k joints have a simultaneous rational parametrization, and the sum of the degree of these parametrizations equals n.

Proof. Let $P \in \mathbb{DH}[t]$ be a motion polynomial of degree n that parametrizes C. The primal part of P cannot have real factors because C is generic. By Theorem 3, there exists a factorization $P = (t - h_1) \cdots (t - h_n)$ with rotation quaternions h_1, \ldots, h_n . Assume that h_i, \ldots, h_{i+m-1} are compatible. Because every dual quaternion compatible with h_i is a real linear combination of 1 and h_i , the product $(t - h_i) \cdots (t - h_{i+m-1})$ can be written

as $R_1 - h_i R_2$ for $R_1, R_2 \in \mathbb{R}[t]$ such that R_1 is monic, $\deg(R_1) = m$, $\deg(R_2) < m$. This already implies the corollary's statement.

Remark 10. It is sometimes advantageous to think of the open kR-chain referred to in Corollary 9 as an open nR-chain with the possibility of coinciding consecutive axes. See, for example, Theorem 11, below.

Example 2. We give a kinematic interpretation of the calculation in Example 1. The two factorizations

$$P = (t - h_1)(t - h_2) = (t - h_1')(t - h_2')$$

show that the motion parametrized by P occurs as end-effector motion of two open 2R-chains, parametrized with the same rational parameter t. The axes of the rotation quaternions h_2 , h_1 , k_1 , k_2 in the moving frame form a closed 4R-chain whose coupler motion is parametrized by P. Hence, we actually presented a method to synthesize a Bennett mechanism to three given poses. Without loss of generality, we assume that one of the poses is the identity. From this data, it is easy to compute the quadratic parametrization of the Bennett motion as in [4]. It serves as input for our factorization algorithm. Note that in general a cubic algebraic number needs to be introduced for the factorization of the real quartic polynomial $P\overline{P}$. This has also been observed in other synthesis algorithms for Bennett linkages.

4 Interchanging factors and closed linkages

In this chapter, we investigate the linkage formed by combining the n! open nR-chains that can be used to generate a generic rational curve C of degree n on the Study quadric.

In order to describe the combinatorial structure of a linkage, we recall the definition of the link graph. Recall that the rigid parts of a linkage are called links, and the existence of a rotational joint between two links means that the two links share a fixed line, the rotation axis. It is possible that m > 2 links share the same rotation axis. In this case, we have $\binom{m}{2}$ different joints supported at this axis. The link graph consists of a node for each link and an edge for each joint connecting two links. This should not be confused with the axis graph, which is also used in the literature. There, the nodes correspond to the axes of joints and the edges correspond to links.

If the linkage moves rationally with mobility one, for every link there exists an algebraic function from \mathbb{P}^1 to \mathbb{P}^7 , mapping a time parameter $t \in \mathbb{P}^1$ to the pose of the link at time t. The pose can be represented by an element of SE₃ or, equivalently, by a point on the Study quadric. If two links L_1 and L_2 are connected by a joint J, the relative motion of L_2 with respect to L_1 is parametrized by the quotient $\psi_J := \phi_1 \phi_2^{-1}$ of the two pose functions. This motion is a rotation. Hence it also has parametrization by a motion polynomial of degree one. The linkage we construct will have the property that ψ_J is a linear polynomial $(t - h_J)$.

Note that when h_J is specified for all joints J, the relative motions of any pair of links can be obtained by multiplication; the linkage kinematics is fully determined. It is easy

to extract the Denavit-Hartenberg parameters from the rotation quaternions h_J . In this paper we are content with the linkage specification by dual quaternions.

Theorem 11. Let $C \subset S$ be a generic rational curve of degree n in the Study quadric, passing through 1. Then C is contained in the motion of a link in a mechanism with revolute joints, 2^n links and $n2^{n-1}$ joints. The link graph is the 1-skeleton of the n-dimensional hypercube.

Remark 12. In Theorem 11 we can only state that C is contained in the motion of one link. It cannot be excluded that the mechanism has more than one degree of freedom. An extreme case arises from the product $P = (t - h_1) \cdots (t - h_n)$ of compatible rotation quaternions h_1, \ldots, h_n . With the understanding of Remark 10, the corresponding link graph can still be considered as 1-skeleton of a hypercube but the linkage has n trivial degrees of freedom. A sufficient condition that ensures equality of C and the link motion is that every 4-cycle in the link graph corresponds to a non-degenerate Bennett linkage.

Proof of Theorem 11. Let P(t) be a generic motion polynomial of degree n, parametrizing C. We denote the n irreducible quadratic factors of $P\overline{P}$ by M_1, \ldots, M_n and construct a linkage whose nodes are labelled by the subsets of $G := \{1, \ldots, n\}$. Two nodes are connected by an edge if and only if the corresponding subsets differ by exactly one element (Figure 1). This graph is known as Hamming graph.

Let $F \subseteq G$ be a subset of cardinality $m \le n$. By successively dividing out right factors $(t-h_i)$ with $M_i = (t-h_i)(t-\overline{h_i}) \in F$, we obtain a factorization P = UV with monic $U, V \in \mathbb{DH}[t]$, and $V\overline{V}$ is the product of the factors in F. This can be done in m! ways. We claim that the result is always the same. To show this, we assume that we have two factorizations $P = U_1V_1 = U_2V_2$ corresponding to two different permutations σ_1, σ_2 of M_1, \ldots, M_n . In our situation, the first m elements in σ_1 are a permutation of the first m elements in σ_2 , in fact this is the set F, and we assume that the remaining elements are equal. This is possible because we are free to choose any order in the remaining factors in $G \setminus F$. Now we apply our non-deterministic algorithm of dividing out right factors to \overline{P} , in the reverse order of the permutations. We obtain the two factorizations $\overline{P} = \overline{V_1U_1} = \overline{V_2U_2}$. But in these two division processes, the first n-m choices are equal, and it follows that $\overline{U_1} = \overline{U_2}$. Consequently, we get $U_1 = U_2$ and $V_1 = V_2$. Hence F determines the right factor V (and also the left factor U) uniquely, and we place the link L_F at position V(t), depending on one real parameter t but not on an ordering of F. This already proves the claim on the linkage graph.

Now we show that linkage consists of revolute joints only. Assume that F_1 and F_2 differ by a single element M_i . Without loss of generality we may assume $F_2 = F_1 \cup \{M_i\}$. Let $P = U_1V_1 = U_2V_2$ be the factorizations obtained as above. Then we have $V_2 = (t - h_i)V_1$, where h_i is unique common solution of U_1 and M_i (see Lemma 6). Hence the relative position of L_{F_2} with respect to L_{F_1} , depending on t, is a rotation group parametrized by $(t - h_i)_{t \in \mathbb{P}^1}$. So we see that L_{F_2} is connected with L_{F_1} by a rotational joint.

The two links whose relative movement is the curve C are easy to determine: They are labelled by the empty set and by G.

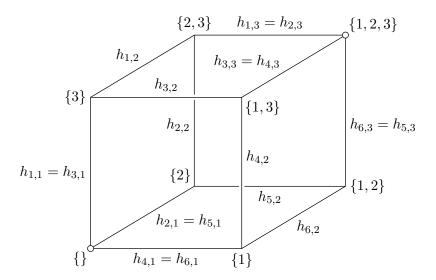


Figure 1: Linkage graph obtained by factorizing a cubic curve

Example 3. We present the construction of a linkage to the graph of Figure 1. A parametrization of a rational cubic curve $C \subset S$ reads

$$P = t^3 - t^2(3 + (4 - \epsilon)\mathbf{i} + (1 + 3\epsilon)\mathbf{j} + 2(1 + \epsilon)\mathbf{k}) - t(3(1 + \epsilon) - 3(3 - \epsilon)\mathbf{i} - (1 + 11\epsilon)\mathbf{j} - (3 + 5\epsilon)\mathbf{k}) + 2(3 - (1 - 3\epsilon)\mathbf{i} - (1 + 2\epsilon)\mathbf{j} - (1 + \epsilon)\mathbf{k}).$$

The real polynomial $P\overline{P}$ factors as

$$P\overline{P} = (\underbrace{t^2 - 2t + 2}_{=:M_1})(\underbrace{t^2 - 2t + 4}_{=:M_2})(\underbrace{t^2 - 2t + 6}_{=:M_3}).$$

The polynomial P admits six factorizations $P = (t - h_{l,1})(t - h_{l,2})(t - h_{l,3})$, one for each permutation of the triple (M_1, M_2, M_3) :

$$(M_1, M_2, M_3): \quad h_{1,1} = 1 + \left(\frac{65}{31} - \frac{814}{961}\epsilon\right)\mathbf{i} + \left(\frac{16}{31} + \frac{1373}{961}\epsilon\right)\mathbf{j} + \left(\frac{18}{31} + \frac{1719}{961}\epsilon\right)\mathbf{k}$$

$$h_{1,2} = 1 + \left(\frac{395}{403} - \frac{94035}{162409}\epsilon\right)\mathbf{i} + \left(\frac{319}{403} + \frac{53380}{162409}\epsilon\right)\mathbf{j} + \left(\frac{41995}{162409}\epsilon + \frac{479}{403}\right)\mathbf{k}$$

$$h_{1,3} = 1 + \left(\frac{12}{13} + \frac{72}{169}\epsilon\right)\mathbf{i} + \left(\frac{210}{169}\epsilon - \frac{4}{13}\right)\mathbf{j} + \left(\frac{3}{13} - \frac{8}{169}\epsilon\right)\mathbf{k}$$

$$(M_1, M_2, M_3): \quad h_{1,1} = 1 + \left(\frac{65}{31} - \frac{814}{961}\epsilon\right)\mathbf{i} + \left(\frac{16}{31} + \frac{1373}{961}\epsilon\right)\mathbf{j} + \left(\frac{18}{31} + \frac{1719}{961}\epsilon\right)\mathbf{k}$$

$$h_{1,2} = 1 + \left(\frac{395}{3403} - \frac{94035}{162409}\epsilon\right)\mathbf{i} + \left(\frac{319}{403} + \frac{53380}{162409}\epsilon\right)\mathbf{j} + \left(\frac{41995}{162409}\epsilon + \frac{479}{403}\right)\mathbf{k}$$

$$h_{1,3} = 1 + \left(\frac{12}{13} + \frac{72}{169}\epsilon\right)\mathbf{i} + \left(\frac{210}{169}\epsilon - \frac{4}{13}\right)\mathbf{j} + \left(\frac{3}{13} - \frac{8}{169}\epsilon\right)\mathbf{k}$$

$$(M_1, M_3, M_2): \quad h_{2,1} = 1 + \left(\frac{5}{3} - \frac{5}{9}\epsilon\right)\mathbf{i} + \left(\frac{1}{3} + \frac{11}{9}\epsilon\right)\mathbf{j} + \left(\frac{14}{9}\epsilon + \frac{1}{3}\right)\mathbf{k}$$

$$h_{2,2} = 1 + \left(\frac{53}{39} - \frac{1324}{1521}\epsilon\right)\mathbf{i} + \left(\frac{38}{39} + \frac{814}{1521}\epsilon\right)\mathbf{j} + \left(\frac{748}{1521}\epsilon + \frac{56}{39}\right)\mathbf{k}$$

$$h_{2,3} = 1 + \left(\frac{12}{13} + \frac{72}{169}\epsilon\right)\mathbf{i} + \left(\frac{210}{169}\epsilon - \frac{4}{13}\right)\mathbf{j} + \left(\frac{3}{13} - \frac{8}{169}\epsilon\right)\mathbf{k}$$

$$(M_2, M_1, M_3): \quad h_{3,1} = 1 + \left(\frac{65}{31} - \frac{814}{961}\epsilon\right)\mathbf{i} + \left(\frac{16}{31} + \frac{1373}{961}\epsilon\right)\mathbf{j} + \left(\frac{1719}{961}\epsilon + \frac{18}{31}\right)\mathbf{k}$$

$$h_{3,2} = 1 + \left(\frac{72}{217} - \frac{16813}{47089}\epsilon\right)\mathbf{i} + \left(\frac{136}{217} - \frac{7695}{47089}\epsilon\right)\mathbf{j} + \left(\frac{14752}{47089}\epsilon + \frac{153}{217}\right)\mathbf{k}$$

$$h_{3,3} = 1 + \left(\frac{11}{7} + \frac{10}{49}\epsilon\right)\mathbf{i} - \left(\frac{1}{7} - \frac{85}{49}\epsilon\right)\mathbf{j} + \left(\frac{5}{7} - \frac{5}{49}\epsilon\right)\mathbf{k}$$

$$(M_2, M_3, M_1): \quad h_{4,1} = 1 + \mathbf{i} + \epsilon\mathbf{j} + \epsilon\mathbf{k}$$

$$h_{4,2} = 1 + \left(\frac{10}{7} - \frac{59}{49}\epsilon\right)\mathbf{i} + \left(\frac{8}{7} + \frac{13}{49}\epsilon\right)\mathbf{j} + \left(\frac{54}{49}\epsilon + \frac{9}{7}\right)\mathbf{k}$$

$$h_{4,3} = 1 + \left(\frac{11}{7} + \frac{10}{49}\epsilon\right)\mathbf{i} - \left(\frac{1}{7} - \frac{85}{49}\epsilon\right)\mathbf{j} + \left(\frac{5}{7} - \frac{5}{49}\epsilon\right)\mathbf{k}$$

$$(M_3, M_1, M_2): \quad h_{5,1} = 1 + \left(\frac{5}{3} - \frac{5}{9}\epsilon\right)\mathbf{i} + \left(\frac{1}{3} + \frac{11}{9}\epsilon\right)\mathbf{j} + \left(\frac{14}{9}\epsilon + \frac{1}{3}\right)\mathbf{k}$$

$$h_{5,2} = 1 + \left(\frac{1}{3} - \frac{4}{9}\epsilon\right)\mathbf{i} + \left(\frac{2}{3} - \frac{2}{9}\epsilon\right)\mathbf{j} + \left(\frac{2}{3} + \frac{4}{9}\epsilon\right)\mathbf{k}$$

$$h_{5,3} = 1 + 2\mathbf{i} + 2\epsilon\mathbf{j} + \mathbf{k}$$

$$(M_3, M_2, M_1): \quad h_{6,1} = 1 + \mathbf{i} + \epsilon\mathbf{j} + \epsilon\mathbf{k}$$

$$h_{6,2} = 1 + (1 - \epsilon)\mathbf{i} + \mathbf{j} + (1 + \epsilon)\mathbf{k}$$

$$h_{6,3} = 1 + 2\mathbf{i} + 2\epsilon\mathbf{j} + \mathbf{k}$$

The rotation quaternions $h_{l,i}$ correspond to edges of the linkage graph in Figure 1. The six rotation quaternions $h_{l,2}$ are all different. The six rotation quaternions $h_{l,1}$ come in three pairs, corresponding to permutations of the shape (M_i, M_j, M_k) and (M_j, M_i, M_k) . Similarly, equal rotation quaternions $h_{l,3}$ come from permutations (M_i, M_j, M_k) and (M_i, M_k, M_j) , respectively.

Remark 13. There exist cases where two consecutive rotation quaternions, like $h_{6,1}$ and $h_{6,2}$ in the above example, are compatible. In this case, the linkage graph of Figure 1 is still correct but with the understanding that the revolute axes to $h_{6,1}$ and $h_{6,2}$ coincide (compare Remark 10). It is possible to reflect this in the linkage graph by removing the edge labeled $h_{6,2}$ and inserting an additional edge connecting $\{\}$ and $\{1,2\}$. The relative motion between the the links $\{\}$ and $\{1,2\}$ is then the product of two linearly parametrized rotations about the same axis, that is, a quadratically parametrized rotation.

From the description of the linkage graph given in Theorem 11 we can draw even more conclusions:

- Every 4-cycle in the mechanism corresponds to a Bennett linkage (general case), to a planar or spherical 4-bar linkage with a rational coupler curve, or to four joints sharing a common revolute axis.
- Any of the n! paths of length n that connects the base $\{\}$ with the platform $\{1,\ldots,n\}$ corresponds to an open nR-chain. Just as any two permutations can be transformed into each other by transpositions, any two of these nR-chains are connected by a series of Bennett substitutions, where two consecutive revolute axes $H_{l,i}$, $H_{l,i+1}$ are replaced by two axes $H_{m,i}$, $H_{m,i+1}$ that complete the first axis pair to the axis quadruple of a Bennett mechanism.
- The mechanism's theoretical degree of freedom, computed according to the formula of Chebychev-Grübler-Kutzbach, is $f_n = 6(2^n 1) 5n2^{n-1}$. Thus, $f_2 = -2$, $f_3 = -18$, $f_4 = -70$, $f_5 = -214$ etc. with rapid decrease as n goes to ∞ .

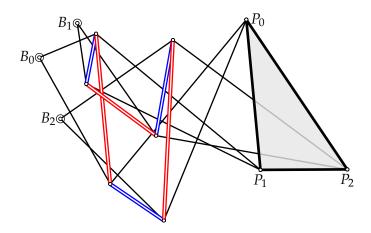


Figure 2: Planar overconstrained linkage

• Our construction stays within the planar motion group SE(2) or the spherical motion group SO(3). That is, if the polynomial P describes a planar or spherical motion, all rotation quaternions obtained from different factorizations of P belong to the same group. Thus, we can construct planar or spherical linkages with the same combinatorics as in the spatial case but with a theoretical degree of freedom of $g_n = 3(2^n - 1) - n2^n$, that is $g_2 = 1$, $g_3 = -3$, $g_4 = -19$, $g_5 = -67$ etc. One example is depicted in Figure 2. It can be thought of as a linkage which is composed of two planar 3RRR-platforms with identical anchor points on the base (B_0, B_1, B_2) and on the platform (P_0, P_1, P_2) . In addition, there exists a hexagon formed by the middle revolute joints, whose side lengths remain constant during the motion. In Figure 2, the hexagon sides are drawn as double lines. An animation of this linkage can be found on the accompanying website http://geometrie.uibk.ac.at/schroecker/qf/. Its theoretical degree of freedom is -3.

5 Translation quaternions and prismatic joints

Theorem 11 admits a small but interesting generalization that allows for a mixture of translational and rotational joints. In fact, the factorization Algorithm 1 also works under slightly more general assumptions: We can allow that the norm polynomial $P\overline{P}$ has a factorization into n distinct monic quadratic polynomials M_1, \ldots, M_n , each of which is either irreducible or the square of a linear polynomial. In this case, the assumptions of Lemma 6 are fulfilled and we again get n! factorizations of P into linear motion polynomials.

If M is a square factor of the norm polynomial $P\overline{P}$, the factor (t-h) satisfying $M=(t-h)(t-\overline{h})$ parametrizes a translational one-parameter subgroup. The primal part of h is a scalar, say λ , and $M=(t-\lambda)^2$; the dual part of H is purely vectorial and specifies the direction of the translation.

If the motion polynomial P has a factorization $P\overline{P} \in \mathbb{R}[t]$ as $P = (t - h_1) \cdots (t - h_n)$ such that the minimal polynomials M_1, \ldots, M_n of h_1, \ldots, h_n are all distinct, we can find this factorization by Algorithm 1, regardless whether h_1, \ldots, h_n represent rotations or translations. As in the purely rotational case, there will be exactly n! such factorizations.

If the factorization $P = (t - h_1) \cdots (t - h_n)$ contains a translation polynomial h_i with primal part p_i , the point $P(p_i)$ lies in the exceptional three-space E. Conversely, if the curve C intersects E and admits a factorized parametrization $P = (t - h_1) \cdots (t - h_n)$ with rotation or translation polynomials h_1, \ldots, h_n , primal(P) must vanish for some parameter value p. This implies that one of the factors $(p - h_1), \ldots, (p - h_n)$ has zero primal part, that is, primal $(h_i) = p \in \mathbb{R}$ and h_i is a translation polynomial.

We illustrate this at hand of two examples.

Example 4. Assume that $P(t) = (t - h_1) \cdots (t - h_n)$ and all quaternions h_1, \ldots, h_n are translation quaternions. This means, that the motion is a pure translation which can be described by the sum of vectors $s(t) = (u_1 + tv_1) + \cdots + (u_n + tv_n)$ with $u_i, v_i \in \mathbb{R}^3$. But s(t) also equals the sum of any permutation of the summands $u_i + tv_i$. This trivial statement is a special case of Theorem 8.

Example 5. The polynomial

$$P = t^2 - t(2 + (1 - \epsilon)\mathbf{i} + (1 + \epsilon)\mathbf{j} + (1 + 2\epsilon)\mathbf{k}) + 1 - 2\epsilon + (1 - \epsilon)\mathbf{i} + (1 + 2\epsilon)\mathbf{j} + (1 + \epsilon)\mathbf{k}$$

admits the factorizations $P = (t - h_1)(t - h_2) = (t - k_1)(t - k_2)$ where

$$h_1 = 1 + \frac{1}{3}((3 - 7\epsilon)\mathbf{i} + (3 + 2\epsilon)\mathbf{j} + (3 + 5\epsilon)\mathbf{k}), \quad h_2 = 1 + \frac{\epsilon}{3}(4\mathbf{i} + \mathbf{j} + \mathbf{k})$$
$$k_1 = 1 + \epsilon\mathbf{j} + \epsilon\mathbf{k}, \quad k_2 = 1 + (1 - \epsilon)\mathbf{i} + \mathbf{j} + (1 + \epsilon)\mathbf{k}.$$

Both, k_1 and k_2 are translation quaternions. Indeed, $P(1) = \epsilon(-2 + \mathbf{j} + \mathbf{k}) \in E$, that is, the curve C with parametrization P intersects the exceptional three-space E.

The kinematic interpretation of Example 5 is well-known. The motion C is the coupler motion of an RPRP linkage [14] and can be obtained as the limiting case of the Bennett motion The observations made in Example 5 immediately generalize to higher degree polynomials.

6 New overconstrained 6R-chains

For n=3, the mechanism described in the preceding section contains a number of overconstrained 6R-chains, some of which are new. This is also true for certain examples in case of n=4. Although only a side result, the discovery of new overconstrained 6R-chains is an important contribution of this article. Therefore we present a more detailed discussion.

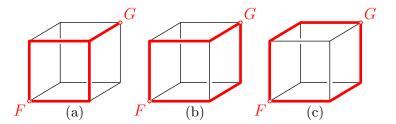


Figure 3: Different types of closed 6R-chains: Bennett linkage plus dangling link (trivial) (a), Waldron's double Bennett hybrid (b), and new (c)

6.1 Coupler curves of degree three

Here, we assume n=3. The link graph is the 1-skeleton of a three-dimensional cube. The relative motion between the link labeled by the empty set $F=\{\}$ (the base) and the link labeled $G=\{1,2,3\}$ (the platform) is a rational cubic curve $C\subset S$ in the Study quadric. In general, there is nothing special about links F and G. We could as well take any two links which are diagonally opposite in the cube.

Every path of length three that connects F and G corresponds to an open 3R-chain which is capable of performing the motion C. Combining two of these paths yields an overconstrained 6R-chain with coupler motion C. There is a total of six paths of lengths three connecting F and G and a total of 15 possible combinations of two such paths. They can be classified into three types of different combinatorics (Figure 3) or, equivalently, into different types of pairs of permutations. We say that two permutations σ_1 and σ_2 differ by $\sigma_2 \circ \sigma_1^{-1}$.

- The first type (a) corresponds to permutations which differ by a neighbor transposition. The resulting 6R-chain is trivial. It consists of a Bennett linkage plus one dangling link. The complete mechanism contains six sub-linkages of this type.
- The second type (b) corresponds to permutations that differ by a cyclic permutation. The resulting 6R-chain is Waldron's double Bennett hybrid [15, pp. 63–65]. There are six sub-linkages of this type.
- The third type (c) is new. It corresponds to permutations that differ by the permutation (3, 2, 1). Three of the 15 6R-chains are of this type.

Currently, overconstrained 6R-chain are classified by relations between their Denavit-Hartenberg parameters (distance, angle, and offset) on the cyclic sequence of revolute axes, see for example the list at the end of [16]. The reasons for their mobility are of geometric nature. Often, they are inferred from the mobility of related structures that are already known to be flexible. Our newly found examples are mobile for *algebraic reasons* or, more precisely, our construction and proof of mobility is algebraic. Therefore, we do not offer simple relations between their Denavit-Hartenberg parameters. However, it is possible to compute exact symbolic expressions of these parameters for numeric examples.

Below we list the angles, distances and offsets for the three new overconstrained 6R-chains obtained from Example 3.

Linkage $h_{1,3},\ h_{1,2},\ h_{1,1},\ \overline{h}_{6,1},\ \overline{h}_{6,2},\ \overline{h}_{6,3}$ Distances: $\frac{16\sqrt{29}}{377},\ \frac{\sqrt{1115179082}}{63302},\ \frac{37\sqrt{854}}{1586},\ \frac{24\sqrt{145}}{899},\ \frac{\sqrt{2}}{2},\ \frac{\sqrt{6}}{6}$ Offsets: $\frac{7945}{59218},\ \frac{38174\sqrt{3}}{62281},\ \frac{545\sqrt{5}}{3538},\ \frac{7}{58},\ \frac{2\sqrt{3}}{3},\ \frac{11\sqrt{5}}{58}$ Angle cosines: $\frac{27\sqrt{5}}{65},\ \frac{29\sqrt{3}}{93},\ \frac{41\sqrt{15}}{195},\ \frac{13\sqrt{5}}{31},\ \frac{\sqrt{3}}{3},\ \frac{\sqrt{15}}{5}$ Linkage $h_{2,3},\ h_{2,2},\ h_{2,1},\ \overline{h}_{4,1},\ \overline{h}_{4,2},\ \overline{h}_{4,3}$ Distances: $\frac{\sqrt{1115179082}}{185822},\ \frac{8\sqrt{29}}{87},\ \frac{37\sqrt{854}}{1586},\ \frac{\sqrt{2}}{6},\ \frac{12\sqrt{145}}{203},\ \frac{\sqrt{6}}{6}$ Offsets: $\frac{7945}{59218},\ \frac{1765\sqrt{5}}{3538},\ \frac{16\sqrt{3}}{61},\ \frac{7}{58},\ \frac{31\sqrt{5}}{58},\ \frac{968\sqrt{3}}{3063}$ Angle cosines: $\frac{151\sqrt{3}}{273},\ \frac{4\sqrt{5}}{15},\ \frac{41\sqrt{15}}{195},\ \frac{5\sqrt{3}}{9},\ \frac{2\sqrt{5}}{7},\ \frac{\sqrt{15}}{5}$ Linkage $h_{3,3},\ h_{3,2},\ h_{3,1},\ \overline{h}_{5,1},\ \overline{h}_{5,2},\ \overline{h}_{5,3}$ Distances: $\frac{\sqrt{6}}{42},\ \frac{\sqrt{1115179082}}{63302},\ \frac{12\sqrt{145}}{203},\ \frac{37\sqrt{854}}{11346},\ \frac{\sqrt{2}}{2},\ \frac{8\sqrt{29}}{87}$ Offsets: $\frac{968\sqrt{3}}{3063},\ \frac{53315}{59218},\ \frac{545\sqrt{5}}{3538},\ \frac{16\sqrt{3}}{61},\ \frac{53}{58},\ \frac{11\sqrt{5}}{58}$ Angle cosines: $\frac{9\sqrt{15}}{35},\ \frac{29\sqrt{3}}{93},\ \frac{2\sqrt{5}}{7},\ \frac{359\sqrt{15}}{1395},\ \frac{\sqrt{3}}{3},\ \frac{4\sqrt{5}}{15}$

6.2 Coupler curves of degree four

Now we consider a second construction of new overconstrained 6R-chains whose coupler curve is of degree four (n=4). Let h_1, h_2, h_3, h_4 be dual quaternions representing rotations. Set $P=(t-h_1)(t-h_2)(t-h_3)(h-h_4)$ and $M_i:=(t-h_i)(t-\overline{h_i})$ for i=1,2,3,4. The link graph of the linkage constructed in Theorem 11 has 16 links and 32 joints. Any open chain connecting base and platform has four links, any closed chain composed of two open chains of this type has eight links. In order to get a 6R-chain, we assume that (h_1,h_2) and (h_3,h_4) are compatible pairs. Now we use Algorithm 1 to compute another factorization $P=(t-g_1)(t-g_2)(t-g_3)(t-g_4)$. This gives rise to an 8R-linkage. But we can remove the link between h_1 and h_2 and the link between h_3 and h_4 in order to get a 6R-linkage. In contrast to the previous examples, the relative motion of some neighbouring links is rational of degree two.

The second factorization depends on the chosen permutation of the factors M_i , i = 1, ..., 4. Of course, using the permutation (M_1, M_2, M_3, M_4) is prohibited. Because of compatibility relations, we also have

$$P = (t - h_2)(t - h_1)(t - h_3)(t - h_4) = (t - h_1)(t - h_2)(t - h_4)(t - h_3).$$

Hence, we must avoid common right or left factors with these two factorizations as they lead to dangling links. In total, there are 16 permutations of this type, because they start with M_1 or M_2 , or end with M_3 or M_4 , or both.

A possible choice of a permutation is (M_3, M_1, M_4, M_2) . The resulting linkage is known as "serial Goldberg 6R linkage" (see [15, 4.3.1]). There are four permutations that lead to this type of linkage.

Another possible choice is (M_3, M_4, M_1, M_2) . The resulting linkage is of a new type and, again, can be obtained from four different permutations. There are some apparent relations between the Denavit-Hartenberg parameters, which can be explained by the fact that the axes of three joints of the 6R-linkage are axes of a 4-cycle of the full "hypercube" linkage. But these relations are not sufficient for the mobility.

7 Conclusion

Factorization of motion polynomials is a new and powerful method for synthesizing linkages with a prescribed rational coupler motion. We gave an explicit construction, based on dual quaternion factorization, for such linkages. It allows to incorporate translation quaternions (prismatic joints) and compatible quaternions (high degree relative rotations of neighboring links).

We studied the resulting linkages and explained how to obtain new types of closed 6R-chains from them. We provided ideas for their construction, but the method leaves room for creativity which may lead to more new types. For example, it is possible to construct overconstrained chains with 2k prismatic and 6-2k revolute joints by combining the ideas of Sections 5 and 6. Coinciding consecutive joints will produce overconstrained chains consisting of five joints (Goldberg linkages).

An important new insight is that flexibility of overconstrained 6R-chains has rather algebraic than geometric reasons. Currently, the authors work on a systematic description and computational tools for synthesizing linkages based on this new method.

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