Classification of Angle-Symmetric 6R Linkages

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Abstract

In this paper, we consider a special kind of overconstrained 6R closed linkages which we call angle-symmetric 6R linkages. These are linkages with the property that the rotation angles are equal for each of the three pairs of opposite joints. We give a classification of these linkages. It turns that there are three types. First, we have the linkages with line symmetry. The second type is new. The third type is related to cubic motion polynomials.

Keywords: Dual quaternion, overconstrained 6R linkages, classification, angle-symmetric

1. Introduction

Movable closed 6R linkages have been considered by many authors (see [1, 2, 3, 4, 5, 6]). In this paper, we give the complete classification of a certain class of such linkages, which we call angle-symmetric. This means that the rotation angles at the three pairs of opposite joints are equal for all possible configurations, or at least for infinitely many configurations (it could be that a certain linkage has two components, where only one of them is angle-symmetric). It is well-known that the line symmetric linkage of Bricard [4] is angle-symmetric. A second family is new; it can be characterized by the presence of three pairs of parallel rotation axes. This fills a gap in [7, Section 3.8]. A third family was discovered in [8, 9] using factorizations of cubic motion polynomials.

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Our main tool is the λ-matrix of a linkage, to be defined in section 2, and its rank r. Intuitively speaking, the configuration set can be described as the vanishing set of r equations in three variables, namely the cotangents of the half of the rotation angles. We will show that r is either 2, 3, or 4. If $r = 2$, then the linkage is line symmetric. If $r = 3$, then we get the new linkage with three pairs of parallel axes. If $r = 4$, then we obtain the linkage described in [8, 9] using motion polynomials.

We use Study's description of Euclidean displacements by the algebra $\mathbb{DH}$ of dual quaternions (see [8, 9]).

Structure of the paper. The remaining part of the paper is set up as follows. In Section 2, we give the definition of the λ-matrix. We also show that the rank of this matrix is 2, 3, or 4. Section 3 contains the main result and examples

2. The $\boldsymbol{\lambda}$-matrix

In this section we define, for a given linkage, a matrix $\mathbf{M}_\lambda$ whose rows are related to an algebraic system defining the configuration space. In the next section, we will see that the rank of this matrix is the basic criterion for classifying angle-symmetric linkages.

The set of all possible motions of a closed 6R linkage is determined by the position of the six rotation axes in some fixed initial configuration. (The choice of the initial configuration among all possible configurations is arbitrary. In some later steps in the classification, we will occasionally change the initial configuration.)

The algebra $\mathbb{DH}$ of dual quaternions is the 8-dimensional real vector space generated by $1, \epsilon, \mathbf{i}, \mathbf{j}, \mathbf{k}, \epsilon\mathbf{i}, \epsilon\mathbf{j}, \epsilon\mathbf{k}$ (see [8, 9]). Following [8, 9], we can represent a rotation by a dual quaternion of the form $\left(\cot\left(\frac{\phi}{2}\right) - h\right)$, where ϕ is the rotation angle and h is a dual quaternion such that $h^2 = -1$ depending only on the rotation axis. We use projective representations, which means that two dual quaternions represent the same Euclidean displacement if only if one is a real scalar multiple of the other.

Let L be a 6R linkage given by 6 lines, represented by dual quaternions $h_1, \ldots, h_6$ such that $h_i^2 = -1$ for $i = 1, \ldots, 6$. A configuration (see [8, 9]) is a 6-tuple $(t_1, \ldots, t_6)$, such that the closure condition

$$(t_1 - h_1)(t_2 - h_2)(t_3 - h_3)(t_4 - h_4)(t_5 - h_5)(t_6 - h_6) \in \mathbb{R}\backslash\{0\}$$

holds. The configuration parameters t_i – the cotangents of the rotation angles – may be real numbers or ∞, and in the second case we evaluate the expression $(t_i - h_i)$ to 1, the rotation with angle 0. The set of all configurations of L is denoted by K_L.

There is a subset of K_L, denoted by K_{sym}, defined by the additional restrictions $t_1 = t_4, t_2 = t_5, t_3 = t_6$. We assume that K_{sym} is a one-dimensional set, i.e. the linkage has an angle-symmetric motion. Mostly, we will assume, slightly stronger, that there exists an irreducible one-dimensional set for which none of the t_i is fixed. Such a component is called a non-degenerate component. We also exclude the case $\dim_{\mathbb{C}} K_{sym} \geq 2$. Linkages with mobility ≥ 2 do exist, but they are well understood.

The closure condition is equivalent to

$$(t_1 - h_1)(t_2 - h_2)(t_3 - h_3) = \lambda(t_3 + h_6)(t_2 + h_5)(t_1 + h_4),$$

where λ is a nonzero real value depending on t_1, t_2, t_3. By taking norm on both sides, we get $\lambda^2 = 1$, i.e. $\lambda = \pm 1$. By multiplying both sides with $(t_1 + h_1)$ from the left and with $(t_1 - h_4)$ from the right, and afterwords dividing by $(t_1^2 + 1)$, we obtain the equation

$$(t_2 - h_2)(t_3 - h_3)(t_1 - h_4) = \lambda(t_1 + h_1)(t_3 + h_6)(t_2 + h_5).$$

Similarly, we obtain

$$(t_3 - h_3)(t_1 - h_4)(t_2 - h_5) = \lambda(t_2 + h_2)(t_1 + h_1)(t_3 + h_6),$$

$$(t_1 - h_4)(t_2 - h_5)(t_3 - h_6) = \lambda(t_3 + h_3)(t_2 + h_2)(t_1 + h_1),$$

$$(t_2 - h_5)(t_3 - h_6)(t_1 - h_1) = \lambda(t_1 + h_4)(t_3 + h_3)(t_2 + h_2),$$

$$(t_3 - h_6)(t_1 - h_1)(t_2 - h_2) = \lambda(t_2 + h_5)(t_1 + h_4)(t_3 + h_3).$$

We may divide K_{sym} into two disjoint subsets K_{sym}^+ and K_{sym}^-, according to whether λ is equal to $+1$ or -1 in the equations above. Any irreducible component of K_{sym} is either contained in K_{sym}^+ or in K_{sym}^-. Note that ∞^3 is an element of K_{sym}^+.

Remark 1. *When we want to study some component $K_0 \subset K_{sym}^-$, we may proceed in the following way: we take a configuration $\tau \in K_0$, which defines a set of rotations around the joint axes. Then we apply these rotations, obtaining new positions for the 6 lines. In the transformed linkage, the component corresponding to K_0 contains ∞^3. So we will always assume that $\lambda = 1$.*

When $\lambda = 1$, after moving the right parts of the above equations to the left, we get an equation

$$\mathbf{M}^{\dagger}\mathbf{X} = \mathbf{0},$$

where $\mathbf{X} = [t_1t_2, t_1t_3, t_2t_3, t_3, t_2, t_1, 1]^T$. If we denote $h_6 + h_3, h_5 + h_2, h_4 + h_1$ by g_3, g_2, g_1 respectively, then the coefficient matrix $\mathbf{M}^{\dagger}$ is

$$\begin{bmatrix} g_3, g_2, g_1, & h_5h_4 - h_1h_2, & h_6h_4 - h_1h_3, & h_6h_5 - h_2h_3, & h_6h_5h_4 + h_1h_2h_3 \\ g_3, g_2, g_1, & h_1h_5 - h_2h_4, & h_1h_6 - h_3h_4, & h_6h_5 - h_2h_3, & h_1h_6h_5 + h_2h_3h_4 \\ g_3, g_2, g_1, & h_2h_1 - h_4h_5, & h_1h_6 - h_3h_4, & h_2h_6 - h_3h_5, & h_2h_1h_6 + h_3h_4h_5 \\ g_3, g_2, g_1, & h_2h_1 - h_4h_5, & h_3h_1 - h_4h_6, & h_3h_2 - h_5h_6, & h_3h_2h_1 + h_4h_5h_6 \\ g_3, g_2, g_1, & h_4h_2 - h_5h_1, & h_4h_3 - h_6h_1, & h_3h_2 - h_5h_6, & h_4h_3h_2 + h_5h_6h_1 \\ g_3, g_2, g_1, & h_5h_4 - h_1h_2, & h_4h_3 - h_6h_1, & h_5h_3 - h_6h_2, & h_5h_4h_3 + h_6h_1h_2 \end{bmatrix}.$$

Note that $\mathbf{M}^{\dagger}$ is a 6×7 matrix with entries in dual quaternions. We also consider $\mathbf{M}^{\dagger}$ to be a 48×7 matrix with real entries. It can be decomposed into submatrices $M_1^{\dagger}, \cdots, M_6^{\dagger}$, where $M_i^{\dagger}$ is the real 8×7 matrix – or the row vector with 7 dual quaternion entries – corresponding to the $i-th$ equivalent formulation of the closure condition above, for $i = 1, \ldots, 6$.

Our classification is based on the following theorem which gives the bounds for the rank of $\mathbf{M}^{\dagger}$.

Theorem 1. *Assume that K_{sym} contains a non-degenerate component of dimension 1. Then $r := \text{rank}(\mathbf{M}^{\dagger}) \in \{2, 3, 4\}$.*

Before we prove Theorem 1, we give a lemma.

Lemma 1. *Assume that K_{sym} contains a non-degenerate component K_0 of dimension 1 such that $\infty^3 \in K_0$, and $r \geq 4$. Then there exists a polynomial of the form*

$$bt_1 + ct_2 + d,$$

where $b, c, d \in \mathbb{R}$ and $bc \neq 0$, which vanishes on K_{sym}, maybe after some permutation of the variables t_1, t_2, t_3. Moreover, we can define a matrix $\mathbf{N}^{\dagger}$ of rank $\geq r - 2$ such that the projection of K_{sym} to (t_1, t_3) is defined by

$$\mathbf{N}^{\dagger}\mathbf{X}' = \mathbf{0}, \tag{1}$$

where $\mathbf{X}' = [t_1^2, t_1t_3, t_1, t_3, 1]^T$.

PROOF. As $r \geq 4$, we have at least four independent equations in three variables (t_1, t_2, t_3) of tridegree at most $(1, 1, 1)$. We denote four of them by F_1, F_2, F_3, F_4.

First, we assume that the F_1 is irreducible. The resultants of F_1 and F_i, $i = 2, 3, 4$ with respect to the last variable t_3 are denoted by F_{12}, F_{13}, F_{14}. The bidegrees of them are at most $(2, 2)$. All these polynomials vanish on K_{sym}. If one of them is 0, such as $F_{12} = 0$, then F_1 and F_2 must have a non-trivial common factor. This can only be F_1, since F_1 is irreducible. Then the tridegree of F_1 is less then $(1, 1, 1)$. Because F_1 vanishes on the non-degenerate component K_0, it must contain at least two variables, and so F_1 is a polynomial of degree $(1, 1, 0)$, maybe after some permutation of variables.

If none of the three resultants vanishes, then let $G = gcd(F_{12}, F_{13}, F_{14})$. The bidegree of G is in the set $\{(2, 2),\ (2, 1),\ (1, 1)\}$, up to permutation of variables t_1, t_2. If it is $(1, 1)$, then G can be considered as a polynomial of tridegree $(1, 1, 0)$ that vanishes on K_0. If the bidegree of G is $(2, 2)$ or $(2, 1)$, then we write $F_{12} = GU_2, F_{13} = GU_3, F_{14} = GU_4$ with suitable polynomials U_2, U_3, U_4. The bidegrees of U_2, U_3, U_4 are at most $(0, 1)$, hence $U_2,\ U_3,\ U_4$ are linear dependent, which means that there are three real number $\lambda_2,\ \lambda_3,\ \lambda_4$ such that

$$\lambda_2 F_{12} + \lambda_3 F_{13} + \lambda_4 F_{14} = 0.$$

As a consequence, we have

$$Res(F_1, \lambda_2 F_2 + \lambda_3 F_3 + \lambda_4 F_4) = 0,$$

where Res denotes the resultant. Then we can continue as in the case $F_{12} = 0$ above. Again we get a polynomial of degree $(1, 1, 0)$, maybe after some permutation of variables.

Second, if F_1 is reducible, then it has two factors with degree $(1, 1, 0)$ and $(0, 0, 1)$, up to permutation of variables t_1, t_2, t_3. Again, F_1 vanishes on the non-degenerate component K_0, and so it must contain at least two variables, and so it is a polynomial of degree $(1, 1, 0)$, maybe after some permutation of variables.

In all cases above, we have a polynomial of tridegree $(1, 1, 0)$ vanishing on K_0. Since ∞^3 is in K_{sym}, it is of the form $bt_1 + ct_2 + d = 0$, with $b, c, d \in \mathbb{R}$ and $bc \neq 0$, as stated in the lemma. We can use it to eliminate t_2: on K_0, we have $t_2 = -\frac{bt_1+d}{c}$.

The equations for the projection of K_0 to the (t_1, t_3)-plane can be obtained by substituting. We get the equation $\mathbf{N}^\dagger \mathbf{X}' = \mathbf{0}$, where $\mathbf{N}^\dagger := \mathbf{M}^\dagger \mathbf{L}$, and

$$\mathbf{L} = \begin{bmatrix} \frac{-b}{c} & 0 & \frac{-d}{c} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{-b}{c} & 0 & \frac{-d}{c} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{-b}{c} & 0 & \frac{-d}{c} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

This follows from the fact that on K_0, we can replace $\mathbf{X}$ by $\mathbf{L}\mathbf{X}'$. Because $\mathrm{rank}(L) = 5$, we also get $\mathrm{rank}(\mathbf{N}^\dagger) \geq \mathrm{rank}(\mathbf{M}^\dagger) - 2$. □

Proof of Theorem 1. $r \geq 2$: Assume, indirectly, that $r \leq 1$. Then the system $\mathbf{M}^\dagger \mathbf{X} = \mathbf{0}$ is equivalent to zero or only one single equation in three variables, and it will have at least a two-dimensional complex configuration set, which contradicts our assumption.

$r \leq 4$: Assume, indirectly, that $r \geq 5$. Then from Lemma 1, the projection of K_{sym} to (t_1, t_3) is defined by

$$\mathbf{N}^\dagger \mathbf{X}' = \mathbf{0}, \tag{2}$$

where $r_1 := \mathrm{rank}(\mathbf{N}^\dagger) \geq r - 2 \geq 3$. The equation (2) is equivalent to a system of r_1 polynomial equations of bidegree at most $(2, 1)$. Because K_{sym} is a curve and has non-degenerate components, the r_1 polynomials have a common factor with bidegree at least $(1, 1)$. Then $r_1 \leq 2$ which contradicts to $r_1 \geq 3$.

3. Classification

This section contains three parts. First, we show that the existence of a line symmetry implies $r = 2$. Second, we show that $r = 2$ or $r = 3$ implies a line symmetry or another geometric consequence which we call the "parallel property". Third, we relate the case $r = 4$ to a linkage described in [8, 9].

3.1. Line Symmetric Linkages

We now describe line symmetric 6R linkages in terms of dual quaternions. A 6R linkage $L = [h_1, h_2, h_3, h_4, h_5, h_6]$ is line symmetric if and only if there is a line represented by a dual quaternion l such that $l^2 = -1$ and

$$h_1 = l h_4 l^{-1}, \quad h_2 = l h_5 l^{-1}, \quad h_3 = l h_6 l^{-1}, \tag{3}$$

where $ll^{-1} = 1$. Geometrically, the rotation around l by the angle π takes h_i to h_{i+3} for $i = 1, 2, 3$.

Proposition 1. *If L is line symmetric, then $r = 2$.*

PROOF. As the norm of l is equal to 1, it follows $l^{-1} = -l$ and we write (3) as

$$h_1 = -lh_4l, \quad h_2 = -lh_5l, \quad h_3 = -lh_6l. \tag{4}$$

We define a map α from dual quaternion to itself as

$$\alpha : \mathbb{DH} \longrightarrow \mathbb{DH}, \qquad h \longmapsto h + l\bar{h}l,$$

where $\bar{h}$ denotes the conjugate of h in dual quaternion. It is true that all entries of $M_1^\dagger$ are in $\mathrm{Im}(\alpha)$. For instance, we have $\alpha(h_1) = h_1 - lh_1l = h_1 + h_4 = g_1, \alpha(h_5h_4) = h_5h_4 + lh_4h_5l = h_5h_4 - (lh_4l)(-lh_5l) = h_5h_4 - h_1h_2, \alpha(h_6h_5h_4) = h_6h_5h_4 - lh_4h_5h_6l = h_6h_5h_4 + (-lh_4l)(-lh_5l)(-lh_6l) = h_6h_5h_4 + h_1h_2h_3$. It is not difficult to prove that α is a $\mathbb{R}$-linear map. If we consider $M_1^\dagger$ to be an 8×7 matrix with real entries, then $r_2 := \mathrm{rank}(M_1^\dagger)$ is less or equal to the dimension of $\mathrm{Im}(\alpha)$. W.l.o.g. we assume $l = \mathbf{i}$. We compute $\mathrm{Im}(\alpha)$ as $\alpha(1) = 1+\mathbf{ii} = 1-1 = 0, \alpha(\epsilon) = \epsilon+\epsilon\mathbf{ii} = 0, \alpha(\mathbf{i}) = \mathbf{i}-\mathbf{iii} = 2\mathbf{i}, \alpha(\mathbf{j}) = \mathbf{j} - \mathbf{iji} = 0, \alpha(\mathbf{k}) = \mathbf{k} - \mathbf{iki} = 0, \alpha(\epsilon\mathbf{i}) = \epsilon\mathbf{i} - \epsilon\mathbf{iii} = 2\epsilon\mathbf{i}, \alpha(\epsilon\mathbf{j}) = \epsilon\mathbf{j} - \epsilon\mathbf{iji} = 0, \alpha(\epsilon\mathbf{k}) = \epsilon\mathbf{k} - \epsilon\mathbf{iki} = 0$. Therefore, the dimension of $\mathrm{Im}(\alpha)$ is 2. So we have $r_2 \leq 2$.

The next step is to prove that all $M_i^\dagger$ for $i = 1, ..., 6$ are equal. It is true that the first three columns are equal in all $M_i^\dagger$ for $i = 1, ..., 6$. As $\mathrm{Im}(\alpha)$ is equal to $\langle \mathbf{i}, \epsilon\mathbf{i}\rangle_{\mathbb{R}}$ and $g_1, g_2, g_3, h_6h_5 - h_2h_3 \in \mathrm{Im}(\alpha)$, we obtain

$$g_1 \times g_2 = g_1 \times g_3 = g_2 \times g_3 = (h_6h_5 - h_2h_3) \times g_1 = 0, \tag{5}$$

where $g \times h$ denotes the cross product of purely vectorial dual quaternions g, h. The equalities $M_1^\dagger = \cdots = M_6^\dagger$ can be shown from (5). For instance, $h_5h_4 - h_1h_2 - (h_1h_5 - h_2h_4) = h_5 \times h_4 - h_1 \times h_2 - h_1 \times h_5 + h_2 \times h_4 = g_2 \times h_4 - h_1 \times g_2 = g_2 \times g_1 = 0, h_1h_5 - h_2h_4 - (h_4h_2 - h_5h_1) = h_1h_5 - h_2h_4 + (\overline{h_1h_5 - h_2h_4}) = 0$ or $h_6h_5h_4 + h_1h_2h_3 - (h_1h_6h_5 + h_2h_3h_4) = -\langle h_6, h_5\rangle h_4 + \langle h_2, h_3\rangle h_4 - \langle h_2, h_3\rangle h_1 + \langle h_6, h_5\rangle h_1 + (h_6 \times h_5) \times h_4 + h_1 \times (h_2 \times h_3) - h_1 \times (h_6 \times h_5) - (h_2 \times h_3) \times h_4 = (h_6 \times h_5 + h_3 \times h_2) \times g_1 = (h_6h_5 - h_2h_3) \times g_1 = 0$, where $\langle g, h\rangle$ denotes the inner product of purely vectorial dual quaternions g, h. As a consequence, we have $r = r_2 \leq 2$. But we have $r \geq 2$ by Theorem 1, so $r = 2$. □

Remark 2. *The well-known fact that line symmetric linkages are movable can also be obtained as a corollary from Theorem 1. When $r = 2$, then the configuration set is defined by 2 equations in 3 variables.*

3.2. Linkages with Rank 2 and 3

In this subsection, we show that $r = 2$ or 3 implies either a line symmetry or another property, defined as follows. We say that $L = [h_1, \ldots, h_6]$ has the parallel property if $h_1 \parallel h_4$, $h_2 \parallel h_3$, $h_5 \parallel h_6$, maybe after some cyclic permutation of indices. In this section, we always assume that the rank of the λ-matrix of L is 2 or 3.

In the following, we use the technique of generic points of algebraic curves. This simplifies the analysis a lot. Let C be an irreducible algebraic curve. Let F be a field such C can be defined by equations over F (for instance $F = \mathbb{Q}$). Following [10, Section 93], we say that some point $p \in C$ is generic if it fulfills no algebraic conditions defined by polynomials with coefficients in F, excerpt those that are a consequence of the equations of C. The existence of generic points is shown in [10, Section 93]; typically, the coordinates of a generic point are transcendental numbers.

Let $K_0 \subset K_{sym}^+$ be an irreducible non-degenerate component of the linkage $L = [h_1, \ldots, h_6]$, and let $\tau_0 = (t_1', t_2', t_3')$ be a generic point of K_0. The configuration τ_0 corresponds to a set of rotations around the joint axes. When we apply these rotations, we get new positions for the 6 lines, and we define the transformed linkage by $L' = [h_1', h_2', h_3', h_4', h_5', h_6']$. Note that L and L' represent really the same linkage, just in different initial positions.

Lemma 2. *If* $\mathrm{primal}(g_1') = 0$*, then L has the parallel property. Here* $\mathrm{primal}(h)$ *denotes the primal part of the dual quaternion h. More precisely, we will have $h_1 \parallel h_4$, $h_2 \parallel h_3$, $h_5 \parallel h_6$, in all configurations in K_0.*

PROOF. Assume that $\mathrm{primal}(g_1') = 0$. The parallelity of the first and fourth axis can be expressed as a set of polynomial equations in the configuration parameters (t_1, t_2, t_3). These equations are fulfilled for the generic point τ_0. By a well-known property of generic points it follows that they are fulfilled for all points in K_0. For this reason, the first and fourth axis are parallel at all position.

Let $S = [p_1, p_2, p_3, p_4, p_5, p_6]$, where $p_i = \mathrm{primal}(h_i')$ for $i = 1, \ldots, 6$. Then S is a spherical linkage with the first and fourth axis coinciding at all positions. We can separate S into two 3R linkages $S_1 = [p_1, p_2, p_3]$ and $S_2 = [p_4, p_5, p_6]$. A 3R linkage is necessarily degenerate: either some angles are constant or some axes coincide. Since t_2 is not a constant in K_0, we obtain $p_2 = \pm p_3$ or $p_1 = \pm p_2$. Since t_3 is not a constant in K_0, we obtain

$p_2 = \pm p_3$ or $p_1 = \pm p_3$. If $p_2 \neq \pm p_3$, then we have $p_1 = \pm p_2$ and $p_1 = \pm p_3$, a contradiction. So we obtain $p_2 = \pm p_3$. Similarly, we also have $p_5 = \pm p_6$.

Therefore, we get a linkage with $h_1' \parallel h_4'$, $h_2' \parallel h_3'$, $h_5' \parallel h_6'$. Since the parallel property is fulfilled for the generic point of the configuration curve, it is fulfilled for all points in K_0. In particular, the original linkage L has the parallel property. □

There is no i such that $g_i' = 0$ for $i = 1, 2, 3$, because if $g_i' = 0$ would be true, then the lines h_i' and h_{i+3}' would be equal; the initial configuration was chosen generically, so the lines h_i and h_{i+3} would be equal for all configurations in K_0, and this is not possible. Moreover, it is not possible that two of g_i for $i = 1, 2, 3$ have 0 primal parts. In order to prove this, we assume indirectly $\mathrm{primal}(g_2') = 0$ and $\mathrm{primal}(g_3') = 0$. By Lemma 2, we get $h_2 \parallel h_5$, $h_3 \parallel h_4$, $h_1 \parallel h_6$ and $h_3 \parallel h_6$, $h_4 \parallel h_5$, $h_1 \parallel h_2$. It follows that L is a planar 6R Linkage which has mobility more than one.

Before the main theorem, we give several lemmas in the following.

Lemma 3. *Let a, b be two purely vectorial dual quaternions. If $a \times b = 0$, then there is a dual number α such that $b = \alpha a$ or $a = \alpha b$, or the primal parts of a and b both vanish.*

PROOF. Straightforward. □

In the next two proofs, we use the following argument from linear algebra. Let $1 \leq i_1 < \cdots < i_r < i_{r+1} < \cdots < i_s \leq 7$ be integers. Let $A := a_1 M_1^\dagger + \cdots + a_6 M_6^\dagger$ be some linear combination of the matrices $M_1^\dagger, \ldots, M_6^\dagger$, where $a_1, \ldots, a_6 \in \mathbb{R}$. If the vector space generated by the columns $(i_1, \ldots, i_s)$ of $M^\dagger$ is already generated by the columns $(i_1, \ldots, i_r)$ of $M^\dagger$, then the vector space generated by the columns $(i_1, \ldots, i_s)$ of A is also generated by the columns $(i_1, \ldots, i_r)$ of A.

Lemma 4. *If $g_3' \times g_1' = g_2' \times g_1' = 0$, then $g_2' \times g_3' = 0$.*

PROOF. We distinguish two cases.

Case I: $\mathrm{primal}(g_1') \neq 0$. By Lemma 3, there exist $\alpha_2, \alpha_3 \in \mathbb{D}$ such that $g_2' = \alpha_2 g_1'$ and $g_3' = \alpha_3 g_1'$, and it follows that $g_2' \times g_3' = 0$.

Case II: $\mathrm{primal}(g_1') = 0$. Then $\mathrm{primal}(g_2') \neq 0$ and $\mathrm{primal}(g_3') \neq 0$. If there exists $\alpha \in \mathbb{D}$ such that $g_3' = \alpha g_2'$, then $g_2' \times g_3' = 0$. Otherwise, g_1' is a dual multiple of g_2' but g_3' is not, so g_1', g_2', g_3' are linearly independent. Then the first three columns generate the column space of $M^\dagger$. By linear algebra, the first three columns of $A := M_1^\dagger + M_4^\dagger - M_3^\dagger - M_6^\dagger$ also generate the column space of A. But

$$A = [0, 0, 0, 0, 2g_3' \times g_1', 2g_3' \times g_2', *] \tag{6}$$

(we do not care about the last entry denoted by $*$), and it follows that $g_2' \times g_3' = 0$. $\square$

Lemma 5. *We have* $g_3' \times g_1' = g_2' \times g_1' = g_2' \times g_3' = 0$.

Proof. Let r_3 be the dimension of the vector space generated by g_1', g_2', g_3'. If $r_3 = 1$, then it follows that $g_3' \times g_1' = g_2' \times g_1' = g_2' \times g_3' = 0$. If $r_3 = 2$ or $r_3 = 3$, then the vector space V generated by the first 6 columns of $M^\dagger$ is already generated by the first three and one of the other three columns.

Assume, for instance, that V is generated by columns $(1, 2, 3, 6)$. By linear algebra, the corresponding columns also generate the space of the first six columns of

$$M_1^\dagger + M_4^\dagger - M_2^\dagger - M_5^\dagger = [0, 0, 0, 2g_2' \times g_1', 2g_3' \times g_1', 0, *].$$

This implies $g_3' \times g_1' = g_2' \times g_1' = 0$, and by Lemma 4, we also get $g_2' \times g_3' = 0$.

If V is generated by columns $(1, 2, 3, 4)$, then the above linear algebra argument shows $g_1' \times g_3' = g_2' \times g_3' = 0$. The equality $g_2' \times g_1' = 0$ follows again from by Lemma 4, applied to the linkage $[h_3, h_4, h_5, h_6, h_1, h_2]$. The third case, when V is generated by columns $(1, 2, 3, 5)$, is also similar. $\square$

Lemma 6. *If* $\operatorname{primal}(g_i') \neq 0$ *for* $i = 1, 2, 3$*, then* L' *is line symmetric.*

Proof. By Lemma 3, there exists a dual quaternion u and invertible dual numbers $\alpha_1, \alpha_2, \alpha_3$ such that $g_i' = \alpha_i u$ for $i = 1, 2, 3$. Let $\beta := u\bar{u} \in \mathbb{D}$. Because the primal part of u is nonzero, the primal part of β is positive, and $\frac{1}{\sqrt{\beta}}$ is defined. We set $l' := \frac{1}{\sqrt{\beta}}u$. Then $l'^2 = -1$ and $g_i'h_i' = h_i'^2 + h_{i+3}'h_i' = {h_{i+3}'}^2 + h_{i+3}'h_1' = h_{i+3}'g_i'$, hence $h_{i+3}' = g_i'h_i'g_i'^{-1} = l'h_i'l'^{-1}$ for $i = 1, 2, 3$. $\square$

Theorem 2. *If* $r = 2$ or 3*, then* L *has a line symmetry or the parallel property.*

Proof. Let $K_0 \subset K_{sym}^+$ be an irreducible non-degenerate component and $\tau_0 = (t_1, t_2, t_3, t_1, t_2, t_3)$ be a generic point of K_0. We get $L' = [h_1', h_2', h_3', h_4', h_5', h_6']$ by applying the rotations specified in τ. By Lemmas 4, 5, and 6, we conclude that L' has a line symmetry or the parallel property. If a line symmetric linkage moves in an angle symmetric way, then the transformed linkage is also angle symmetric. This implies that when L' is line symmetric, then L is also line symmetric. On the other hand, if L' has the parallel property, then parallelity holds for all points in K_0, in particular L has the parallel property. □

Theorem 3. *If $r = 2$, then L is line symmetric.*

Proof. By Theorem 1 and Theorem 2, we may assume that L has and parallel property and $r = 2$. Let $L' = [h'_1, h'_2, h'_3, h'_4, h'_5, h'_6]$ be the linkage transformed by a generic position. We may assume $h'_1 \parallel h'_4$, $h'_2 \parallel h'_3$, $h'_5 \parallel h'_6$. The primal part of g'_1 is 0 and the primal parts g'_2 and g'_3 are not. We define l' as $\frac{1}{\sqrt{g'_2\overline{g'_2}}}g'_2$. Then $l'^2 = -1$. By Lemma 5, we also get $h'_2 = -l'h'_5l'$ and $h'_3 = -l'h'_6l'$ (see also the proof of Lemma 6). Moreover, g'_1 is a real multiple of $\epsilon l'$, and $g'_1h'_1 = h'_4g'_1$. By the last equation, the primal part of $h'_1 + l'h'_4l'$ is zero. The dual part of $h'_1 + l'h'_4l'$ is equal to $u := g'_1 - h'_4 + l'h'_4l'$. The vectorial part of $ul' = g'_1l' - h'_4l' - lh'_4$ vanishes, so u is a multiple of l'. On the other hand, the scalar product of u with l' also vanishes, hence $u = 0$ and $h'_1 = -l'h'_4l'$. It follows that L' and L are same line symmetric. □

In the end of this subsection, we give a construction of angle-symmetric 6R linkage with parallel property. The construction is based on the fact that we have a partially line symmetry taking h_2 to h_5 and h_3 to h_6 (see Lemma 3 and Lemma 5 above).

Construction 1. *(Angle-Symmetric 6R Linkage with Parallel Property)*

I. Choose a rotation axis u such that $u^2 = -1$.

II. Choose another rotation axis h_1 such that $h_1^2 = -1$ and it is perpendicular to u.

III. Choose two parallel rotation axes h_2 and h_3 which are not perpendicular to u such that $h_2^2 = h_3^2 = -1$.

IV. Set $h_4 = -uh_1u + r\epsilon u$, where r is a random real number.

V. Set $h_5 = -uh_2u$ and $h_6 = -uh_3u$.

VI. Our angle-symmetric 6R Linkage with parallel property is $L = [h_1, h_2, h_3, h_4, h_5, h_6]$. □

Example 1. *(Angle-Symmetric 6R Linkage with Parallel Property) We set*

$$
\begin{aligned}
u &= \mathbf{i},\\
h_1 &= -\frac{7}{11}\epsilon\mathbf{i} + \mathbf{j},\\
h_2 &= \left(2\epsilon - \frac{3}{5}\right)\mathbf{i} - \left(\frac{3}{2}\epsilon + \frac{4}{5}\right)\mathbf{j} - \epsilon\mathbf{k},\\
h_3 &= \left(-2\epsilon + \frac{3}{5}\right)\mathbf{i} + \left(\frac{3}{2}\epsilon + \frac{4}{5}\right)\mathbf{j} + 2\epsilon\mathbf{k},\\
r &= \frac{14}{11},\\
h_4 &= \frac{7}{11}\epsilon\mathbf{i} - \mathbf{j},\\
h_5 &= \left(2\epsilon - \frac{3}{5}\right)\mathbf{i} + \left(\frac{3}{2}\epsilon + \frac{4}{5}\right)\mathbf{j} + \epsilon\mathbf{k},\\
h_6 &= \left(-2\epsilon + \frac{3}{5}\right)\mathbf{i} - \left(\frac{3}{2}\epsilon + \frac{4}{5}\right)\mathbf{j} - 2\epsilon\mathbf{k}.
\end{aligned}
$$

It can be seen that the axes of h_1, h_4 are parallel, and the axes of h_2, h_3 and h_5, h_6, respectively, are parallel. Furthermore, the configuration curve contains a non-degenerate component:

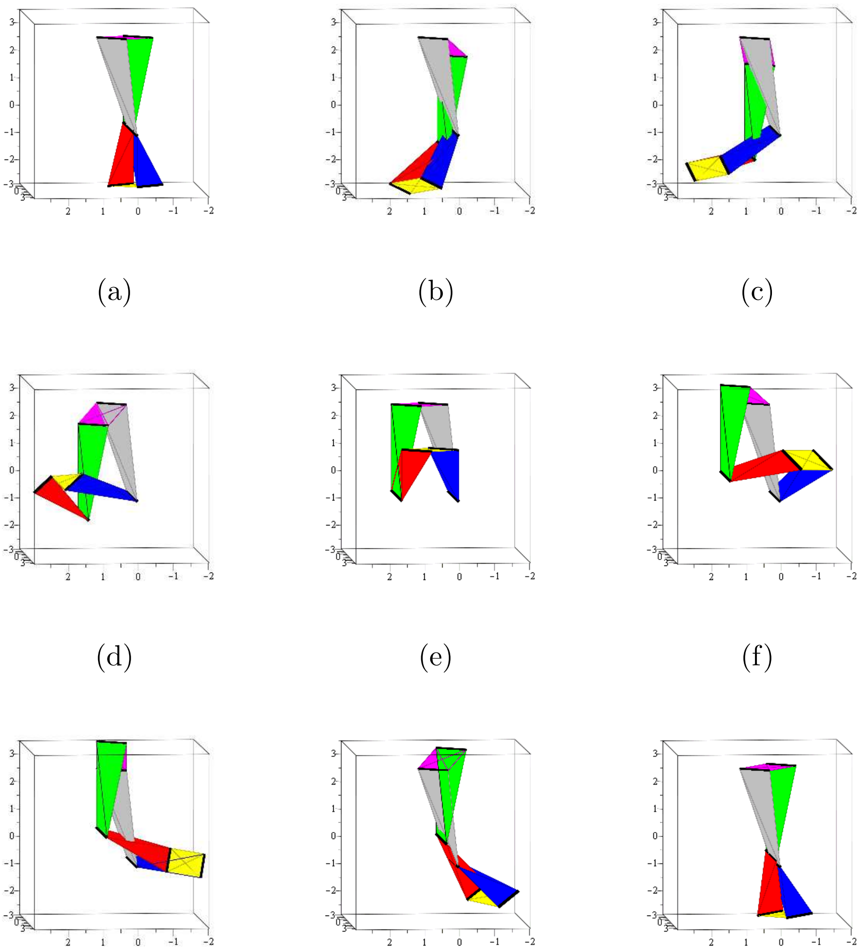
$$(t_1, t_2, t_3, t_4, t_5, t_6) = \left(\frac{5}{4}t, t, t, \frac{5}{4}t, t, t\right).$$

Thus, we have an example of angle-symmetric 6R linkage with parallel property. The rank of $\mathbf{M}^\dagger$ is 3. In Figure 1, we present nine configuration positions of this linkage produced by Maple. □

Remark 3. *A random instance of Construction 1 produces a linkage where t_1 is parametrized by a quadratic function in $t = t_2 = t_3$. This example is special because t_1 is linear in t. (There is a degenerate component of the configuration curve that is responsible for this drop of the degree.)*

3.3. Linkages with Rank 4

In this subsection, we show that the angle-symmetric linkages with Rank 4 are exactly those that have been constructed in [9, Example 3] by factorization of cubic motion polynomials.



(a) (b) (c)

(d) (e) (f)

(g) (h) (i)

Figure 1: These nine pictures which are produced by Maple are different positions of the linkage in Example 1. The six colored tetrahedra(gray, blue, yellow, red, green, pink) represent six links in the linkage, and the joints are common edges of connected tetrahedra.

Recall that a motion polynomial P is a polynomial in one variable t with coefficients in $\mathbb{DH}$ such that $P\bar{P}$ is a real polynomial that does not vanish identically. (Multiplication in $\mathbb{DH}[t]$ is defined by requiring that t commutes with the coefficients in $\mathbb{DH}$.) Motion polynomials parametrize motions: by substituting a real number for t, we obtain an element in the Study quadric.

We give a brief sketch of the construction in [8, 9]. Linear motion polynomials of the form $(t-a-bh)$, $a,b \in \mathbb{R}$, $b \neq 0$, $h \in \mathbb{DH}$, $h^2=-1$ parametrize revolutions. When we multiply three such polynomials R_1, R_2, R_3, we get a cubic motion polynomial Q. Generically, there are 6 different factorizations into linear monic polynomials, and there is one of the form $R_6R_5R_4$ such that the equations $R_1\bar{R}_1 = R_4\bar{R}_4$, $R_2\bar{R}_2 = R_5\bar{R}_5$, $R_3\bar{R}_3 = R_6\bar{R}_6$ hold. The three linear factors R_4, R_5, R_6 are again motion polynomials parametrizing revolutions. The six axes of $R_1, \ldots, R_6$ define a closed 6R linkage; let us call it a linkage of cubic polynomial type.

We set $R_i(t) = t - a_i - b_ih_i$ for $i = 1, \ldots, 6$, $a_i, b_i \in \mathbb{R}$, $b_i \neq 0$, $h_i \in \mathbb{DH}$, $h_i^2 = -1$. The equations above are equivalent to $a_i = a_{i+3}$ and $b_i^2 = b_{i+3}^2$ for $i = 1, 2, 3$. We may even assume $b_i = -b_{i+3}$; if not, we replace h_{i+3} and b_{i+3} by $-h_{i+3}$ and $-b_{i+3}$. We multiply $R_1R_2R_3 = R_6R_5R_4$ by $\bar{R}_4\bar{R}_5\bar{R}_6$ and get that

$$(t-a_1-b_1h_1)(t-a_2-b_2h_2)(t-a_3-b_3h_3)(t-a_1-b_1h_4)(t-a_2-b_2h_5)(t-a_3-b_3h_6)$$

is a real polynomial. This shows that the configuration curve is parametrized by

$$(t_1, t_2, t_3, t_4, t_5, t_6) = \left(\frac{t-a_1}{b_1}, \frac{t-a_2}{b_2}, \frac{t-a_3}{b_3}, \frac{t-a_1}{b_1}, \frac{t-a_2}{b_2}, \frac{t-a_3}{b_3} \right).$$

In particular, the linkage of cubic polynomial type is angle symmetric.

Here is a converse of the above statement.

Theorem 4. *If L is an angle-symmetric linkage such that the λ-matrix has rank $r = 4$, then L is of cubic polynomial type.*

Proof. By Lemma 1, there exist a polynomial of the form $bt_1 + ct_2 + d$ that vanishes on K_{sym}, $b, c, d \in \mathbb{R}$, $bc \neq 0$, and the projection of K_{sym} to (t_1, t_3) is in the common zero set of two linear independent polynomials of bidegree $(2, 1)$. The equation of the projection is therefore a common factor of these two equations and must have bidegree smaller than $(2, 1)$. Since K_{sym} has a non-degenerate component, the common factor cannot be constant in t_1 or t_3, hence it has bidegree $(1,1)$. Because (∞,∞) is contained in the projection, the common factor has the form $b't_1 + c't_2 + d'$ for $b', c', d' \in \mathbb{R}$, $b'c' \neq 0$. This allows to parametrize K_{sym} with linear functions

$$(t_1, t_2, t_3) = \left(\frac{t-a_1}{b_1}, \frac{t-a_2}{b_2}, \frac{t-a_3}{b_3}\right)$$

for $a_1, \dots, b_3 \in \mathbb{R}$, $b_1b_2b_3 \neq 0$. Now the linkage can be reconstructed from the two factorizations of the cubic motion polynomial

$$(t-a_1-b_1h_1)(t-a_2-b_2h_2)(t-a_3-b_3h_3) = (t-a_3+b_3h_6)(t-a_2+b_2h_5)(t-a_1+b_1h_4),$$

so it is of cubic polynomial type. □

4. Conclusion

In the analysis of the case $r = 3$, we obtained a new type of linkages (with parallel property $h_1 \parallel h_4$, $h_2 \parallel h_3$, $h_5 \parallel h_6$). It is not clear from the paper if every linkage with parallel property is angle-symmetric. We know that this is not the case. A complete analysis of linkages with parallel property will be the topic of a future paper.

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