Trackability with Imprecise Localization*

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Abstract

Imagine a tracking agent P who wants to follow a moving target Q in d-dimensional Euclidean space. The tracker has access to a noisy location sensor that reports an estimate $\tilde{Q}(t)$ of the target's true location $Q(t)$ at time t, where $||Q(t)-\tilde{Q}(t)||$ represents the sensor's localization error. We study the limits of tracking performance under this kind of sensing imprecision. In particular, we investigate (1) what is P's best strategy to follow Q if both P and Q can move with equal speed, (2) at what rate does the distance $||Q(t) - P(t)||$ grow under worst-case localization noise, (3) if P wants to keep Q within a prescribed distance L, how much faster does it need to move, and (4) what is the effect of obstacles on the tracking performance, etc. Under a relative error model of noise, we are able to give upper and lower bounds for the worst-case tracking performance, both with or without obstacles.

1   Introduction

The problem of tracking a single known target is a classical one with a long history in artificial intelligence, robotics, computational geometry, graph theory and control systems. The underlying motivation is that many robotic applications including search-and-rescue, surveillance, reconnaissance and environmental monitoring have components that are best modeled as a tracking problem. The problem is often formulated as a *pursuit-evasion* game, with colorful names such as Man-and-the-Lion, Cops-and-Robbers, Hunter-and-Rabbit, Homicidal Chauffeur, and Princess-and-Monster [1, 2, 4, 8]. Visibility-based pursuit evasion [7, 19], in particular, has been a topic of great interest, in part due to its simple but realistic model:

a team of pursuers is tasked with locating a single adversarial evader in an geometric environment with polygonal obstacles where pursuers learn the evader's position only when the latter is in their line-of-sight. After two decades of research, tight bounds are known for detection or capture of the evader for many basic formulations of the problem [3, 7, 11], although the topic remains a rich subject of ongoing research [12, 15].

Most theoretical analyses of tracking, however, assume an idealized sensing model, ignoring the fact that all location sensing is noisy and imprecise in practice: the target's position is rarely known with complete and error-free precision. Although some papers have explored models to incorporate practical limitations of idealized visibility including angular visibility [10], beam sensing [16], field-of-view sensors [6], and range-bounded visibility [5], the topic of sensing noise or imprecision has largely been handled heuristically or through probabilistic techniques such as Kalman filters [9, 14, 18, 20]. One exception is [17], where Rote investigates a tracking problem under the *absolute error* model: in this model, the target's position is always known to lie within distance 1 of its true location, regardless of its distance from the tracker. The analysis in [17] shows that, under this noise model, the distance between the tracker and the target can grow at the rate of $\Theta(t^{1/3})$, where t is the time parameter. Our model, by comparison, deals with a more severe form of noise, with imprecision proportional to the distance from the tracker. In [13], Kuntsevich et al. consider the same relative error model as ours, but without any obstacles. Their work has a control-theoretic perspective, with a primary goal of deriving a bound on the time needed by the tracker to capture the target. Our main contribution is to analyze the worst-case behavior of trackability as a function of the localization precision parameter λ.

Motivation and the Problem Statement. This paper takes a small step towards bridging the gap between theory and practice of trackability, and analyzes the effect of noisy sensing. In particular, we consider a tracking agent P who wants to follow a moving target Q in d-dimensional Euclidean space using a noisy location sensor. For simplicity, we analyze the problem in two dimensions, but the results easily extend to d dimensions, as discussed in Section 5. We use the notation $Q(t)$ and $P(t)$ to denote the (true) positions of the target and the tracker at time t. We adopt a simple but realistic model of *relative* error in sensing noise: the localization error is *proportional* to the true distance between the tracker and the target. More precisely, the localization error is upper bounded as $||Q(t) - \tilde{Q}(t)|| \leq \frac{1}{\lambda}||P(t) - Q(t)||$ at all times t, where $\lambda \geq 1$ is the quality measure of *localization precision*. Thus, the closer the target, smaller the error, and a larger λ means better localization accuracy, while $\lambda = 1$ represents the completely noisy case when the target can be anywhere within a disk of radius $||P(t) - Q(t)||$ around $Q(t)$. *It is important to note that the parameter λ is used only for the analysis, and is not part of information revealed to the tracker.* In other words, the tracker only observes the approximate location $\tilde{Q}(t)$, and *not the uncertainty disk* containing the target. The relative error model is intuitively simple (farther the object, larger the measurement error) and captures the realism of many sensors: for instance, the resolution error in camera-based tracking systems is proportional to the target's distance, and in network-based tracking, *latency* causes a proportionate localization uncertainty because of target's

movement before the signal is received by the tracker.

We study the tracking problem as a game between two players, the tracker P and the target Q, which is played in *continuous time and space*: that is, each player is able to instantaneously observe and react to other's position, and the environment is the two-dimensional plane, with or without polygonal obstacles. Both the target and the tracker can move with equal speed, which we normalize to *one*, without loss of generality. With the unit-speed assumption, the following holds, for all times $t_1 \leq t_2$:

$$||Q(t_2) - Q(t_1)|| \leq |t_2 - t_1|, \quad ||P(t_2) - P(t_1)|| \leq |t_2 - t_1|$$

Under the *relative localization error* model, the reported location of the target $\tilde{Q}(t)$ always satisfies the following bound, where λ is the accuracy parameter:

$$||Q(t) - \tilde{Q}(t)|| \ \leq \ \frac{||P(t) - Q(t)||}{\lambda}$$

We measure the tracking performance by analyzing the distance function between the target and the tracker, namely, $D(t) = d(P(t), Q(t))$, over time, with $D(0)$ being the distance at the beginning of the game. Under error-free localization, the distance remains bounded as $D(t) \leq D(0)$. We analyze how $||D(t) - D(0)||$ grows under the relative error model, as a function of λ. Our main results are as follows.

Our Results. We show that the simple greedy strategy of "always move to the *observed* location of the target" achieves $D(t) \ \leq \ D(0) + t/\lambda^2$. That is, the target's distance from the tracker can grow at most at the rate of $O(\lambda^{-2})$, the inverse quadratic function of the localization parameter. We prove this rate to be worst-case optimal with a matching lower bound: a strategy for the target that ensures that, under the relative error model, it can increase its distance as $D(t) \ \geq \ D(0) + \Omega(t/\lambda^2)$.

We then extend this analysis to environments with polygonal obstacles, and show that the tracker can increase its distance by $\Omega(t)$ in time t *for any finite* λ. This is unsurprising because two points within a small margin of sensing error can be far apart in free-space, thereby fooling the tracker into "blind alleys." More surprisingly, however, if we adopt a localization error that is proportional to the *geodesic* distance (and not the Euclidean distance) between the target and the tracker, then the distance increases at a rate of $\Theta(\lambda^{-1})$. This bound is also tight within a constant factor: the tracker can maintain a distance of $D(t) \ \leq \ D(0) + O(t/\lambda)$ by the greedy strategy, while the target has a strategy to ensure that the distance function grows as at least $D(t) \ \geq \ D(0) + \Omega(t/\lambda)$.

Our analysis also helps answer some other questions related to tracking performance. For instance, a natural way to achieve good tracking performance in the presence of noisy sensing is to let the tracker move at a faster speed than the target. Then, what is the minimum speedup necessary for the tracker to reach the target (or, keep within a certain distance of it)? We derive upper and lower bounds for this speedup function, which are within a constant factor of each other as long as $\lambda \geq 2$.

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Organization. The paper is organized as follows. Section 2 investigates the tracking problem in the Euclidean plane without obstacles. Section 3 derives bounds on the speedup function. Section 4 explores the tracking performance in the presence of polygonal obstacles. Section 5 briefly discussed the straight-forward extension of our results to d-dimensions. Section 6 offers a summary and concluding remarks.

2 Tracking in the Unobstructed Plane

We begin with the simple setting in which a tracking agent P wants to follow a moving target Q in the two-dimensional plane without any obstacles. We show that the trivial "aim for the target's observed location" achieves essentially the best possible worst-case performance. We first prove the upper bound on the derivative $D'(t)$ of the distance function $D(t)$, and then describe an adversary's strategy that matches this upper bound.

2.1 Tracker's Strategy and the Upper Bound

Our tracker uses the following obvious algorithm, whose performance is analyzed in Theorem 1 below.

GREEDYTRACK. At time t, the tracker P moves directly towards the target's observed location $\tilde{Q}(t)$.

Theorem 1. *By using* GREEDYTRACK*, the tracker can ensure that $D(t) \leq D(0) + O(t/\lambda^2)$, for all $t \geq 0$.*

Proof. Consider the true and the observed positions of the target, namely $Q(t)$ and $\tilde{Q}(t)$, respectively, at time t, and let γ be the angle formed by them at $P(t)$. See Figure 1. Consider an arbitrarily small time period Δt during which P moves towards $\tilde{Q}(t)$ and $Q(t)$ moves away from $P(t)$. We want to compute the derivative of the distance function, given as Equation (1).

$$D'(t) \;=\; \lim_{\Delta t \to 0} \frac{D(t + \Delta t) - D(t)}{\Delta t} \tag{1}$$

The new distance between the target and the tracker is given by bc in Fig. 1. In the triangle abc, we have $ab = \Delta t \sin\gamma$ and $ac = D(t) + \Delta t - \Delta t \cos\gamma$. We, therefore, can bound $D(t + \Delta t)$ as follows (where the final inequality uses the fact $\sqrt{1 + x} \leq 1 + \frac{x}{2}$):

$$
\begin{aligned}
D(t + \Delta t) &= \sqrt{(\Delta t \sin\gamma)^2 + (D(t) + \Delta t - \Delta t \cos\gamma)^2} \\
&= \sqrt{\Delta t^2 \sin^2\gamma + D(t)^2 + 2D(t)\Delta t(1 - \cos\gamma) + \Delta t^2 - 2\Delta t^2 \cos\gamma + \Delta t^2 \cos^2\gamma} \\
&= \sqrt{\Delta t^2 + D(t)^2 + 2D(t)\Delta t(1 - \cos\gamma) + \Delta t^2 - 2\Delta t^2 \cos\gamma} \\
&= \sqrt{(D(t) + \Delta t)^2 - 2D(t)\Delta \cos\gamma + \Delta t^2 - 2\Delta t^2 \cos\gamma}
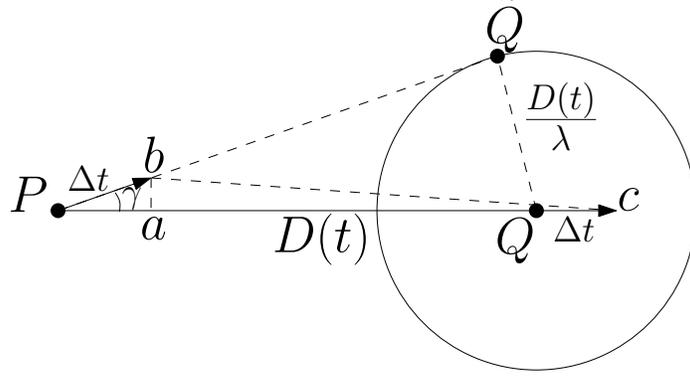\end{aligned}
$$

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Figure 1: Proof of Theorem 1.

$$= \sqrt{(D(t)+\Delta t)^2 - 2\Delta t(D(t)+\Delta t) + \Delta t^2 - 2D(t)\Delta \cos\gamma - 2\Delta t^2 \cos\gamma + 2\Delta t(D(t)+\Delta t)}$$
$$= \sqrt{(D(t)+\Delta t - \Delta t)^2 - 2D(t)\Delta t \cos\gamma - 2\Delta t^2 \cos\gamma + 2\Delta t(D(t)+\Delta t)}$$
$$= \sqrt{D(t)^2 + 2\Delta t(D(t)+\Delta t)(1 - \cos\gamma)}$$
$$= D(t)\sqrt{1 + 2\Delta t(D(t)+\Delta t)(1 - \cos\gamma)/D(t)^2}$$
$$\leq D(t) + (\Delta t)(1 + \Delta t/D(t))(1 - \cos\gamma)$$

Returning to Equation (1), we get

$$D'(t) = \lim_{\Delta t \to 0} \frac{D(t+\Delta t) - D(t)}{\Delta t} \leq \lim_{\Delta t \to 0} (1 + \Delta t/D(t))(1 - \cos\gamma) = 1 - \cos\gamma$$

Finally, since $\sin\gamma \leq \frac{1}{\lambda}$, we get $D'(t) \leq 1 - \sqrt{1 - \frac{1}{\lambda^2}}$, which simplifies by the Taylor series expansion:

$$D'(t) \leq 1 - (1 - \frac{1}{2\lambda^2} - \frac{1}{8\lambda^4} - \cdots) = \frac{1}{2\lambda^2} + \frac{1}{8\lambda^4} + \cdots \leq \frac{1}{\lambda^2}$$

This completes the proof that $D(t) \leq D(0) + t/\lambda^2$. $\qquad\square$

2.2 Target's Strategy and the Lower Bound

We now show that this bound is asymptotically tight, by demonstrating a strategy for the target to grow its distance from the tracker at the rate of $D(t) \geq D(0) + \Omega(t/\lambda^2)$, for all $t \geq 0$. We think of the target as an adversary who can choose its observed location at any time subject only to the constraints of the error bound: $||Q(t) - \tilde{Q}(t)|| \leq \frac{1}{\lambda}(||P(t) - Q(t)||)$. (Recall that the tracker only observes the location $\tilde{Q}(t)$, and has no direct knowledge of either the parameter λ or the distance $||P(t) - Q(t)||$. Those quantities are only used in the analysis. However, the lower bound holds even if the tracker knows the uncertainty disk, namely, the localization error $\frac{1}{\lambda}(||P(t) - Q(t)||)$.)

In order to analyze the lower bound, we divide the time into *phases*, and show that the distance from the tracker increases by a *multiplicative factor* in each phase, resulting in a growth rate of $\Omega(1+\lambda^{-2})$. If the ith phase begins at time t_i, then we let $d_i = ||Q(t_i) - P(t_i)||$ denote the distance between the target and the tracker at t_i. During the ith phase, the target maintains the following invariant for a constant $0 < \alpha < 1$ to be chosen later.

Gap Invariant. Throughout the ith phase, the target moves along a path $Q(t)$ such that $||Q(t) - P(t)|| \geq \alpha d_i$, for all times t, and all reported locations satisfy $||Q(t) - \tilde{Q}(t)|| \leq \alpha d_i/\lambda$.

See Figure 2(i) for an illustration. Consider the isosceles triangle with vertices at $Q(t_i)$, q_a and q_b, whose base $q_a q_b$ is perpendicular to the line $P(t_i)Q(t_i)$. The equal sides of the triangle have length $2d_i$, the base has length $2\alpha d_i/\lambda$, and let q_c be the midpoint of the base. The target's strategy is to move from $Q(t_i)$ to either q_a or q_b, and *report its location $\tilde{Q}(t)$ at the closest point on the line $Q(t_i)q_c$*; i.e. at all times, $\tilde{Q}(t)$ is the perpendicular projection of $Q(t)$ onto the line $Q(t_i)q_c$. By the symmetric construction, and the choice of the points q_a and q_b, the tracker cannot tell whether the target is moving to q_a or q_b. Thus, any deterministic tracker makes an incorrect choice in one of the two possible scenarios. For the *worst-case performance bound*, we can equivalently assume that the target *non-deterministically guesses* the tracker's intention, and moves to the better of the two possible locations, q_a or q_b. The tracker makes this choice based on whether the tracker is on or below the line $Q(t_i)q_c$, or not. In the former case, the target moves to q_a, and to to q_b otherwise. The ith phase terminates when the target reaches either q_a or q_b, and the next phase begins. (We note that, after i phases, there are 2^i possible choices made by the tracker, reflected in whether it is above or below the line $Q(t_i)q_c$ at the conclusion of each phases. For each of these possible "worlds" there is a corresponding deterministic strategy of the target that "fools" the tracker in every phase, resulting in the maximum distance increase.) There is one subtle point worth mentioning here. It is possible that during the phase, the distance between the players may shrink if the tracker temporarily moves towards the same final location as the target—however, our Gap Invariant ensures that that the target's noisy location remains within the permissible error bound throughout the phase. The following lemma shows that this simple strategy of the target can maintain the Gap Invariant for any choice of $\alpha \leq 0.927$.

Lemma 1. *The target can maintain the Gap Invariant for any $\alpha \leq 0.927$.*

Proof. Consider an arbitrary phase i. By construction, we have $||Q(t) - \tilde{Q}(t)|| \leq \frac{\alpha d_i}{\lambda}$ throughout this phase, so we only need to show $D(t) \geq \alpha d_i$. There is one subtle point worth mentioning here. While the target's strategy will ensure that its distance from the tracker grows by a certain multiplicative factor *at the end of the phase*, the distance between the players may shrink during the phases. This happens when the tracker temporarily moves towards the same final location as the target. In spite of this temporary "lucky" guess by the tracker, we need to ensure that the target's noisy location remains within the permissible error bound throughout the phase. The constant α is introduced precisely to guarantee this validity, and we arrive at its value as follows.
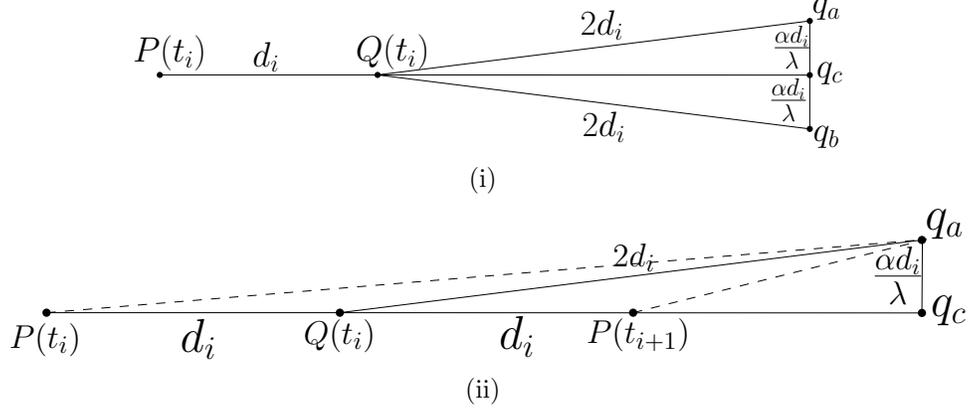
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Figure 2: Target's strategy during the ith phase (i), and proofs of Lemmas 1 and 2 (ii).

Let d_{i+1} be the distance between P and Q if both moved toward q_a for the duration of phase i. Note that d_{i+1} is the length of the segment $P(t_i)q_a$ minus $2d_i$, as shown in Figure 2(ii). The length of $P(t_i)q_a$ can be calculated from the right triangle $q_aP(t_i)q_c$, while the length of q_aq_c is known by construction. Finally, $Q(t_i)q_c$ has length d_i less than $P(t_i)q_c$. Thus, we have:

$$
\begin{aligned}
d_{i+1} &= \sqrt{\left(\sqrt{4d_i^2 - \frac{d_i^2\alpha^2}{\lambda^2}} + d_i\right)^2 + \frac{d_i^2\alpha^2}{\lambda^2}} - 2d_i \\
&= \sqrt{5d_i^2 - \frac{d_i^2\alpha^2}{\lambda^2} + 2d_i\sqrt{4d_i^2 - \frac{d_i^2\alpha^2}{\lambda^2}} + \frac{d_i^2\alpha^2}{\lambda^2}} - 2d_i \\
&= d_i\sqrt{5 + 4\sqrt{1 - \frac{\alpha^2}{4\lambda^2}}} - 2d_i
\end{aligned}
$$

In order to satisfy the Gap Invariant, we must choose an α such that the following inequality holds:

$$
\begin{aligned}
\alpha d_i &\le d_i\sqrt{5 + 4\sqrt{1 - \frac{\alpha^2}{4\lambda^2}}} - 2d_i \\
\alpha^2 + 4\alpha + 4 &\le 5 + 4\sqrt{1 - \frac{\alpha^2}{4\lambda^2}} \\
\alpha^2 + 4\alpha + 4 &\le 5 + 4 - \frac{\alpha^2}{2\lambda^2} \\
\alpha^2\left(1 + \frac{1}{2\lambda^2}\right) + 4\alpha - 5 &\le 0
\end{aligned}
$$

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This gives the following upper bound:

$$\alpha \leq \frac{-4 \pm \sqrt{16 + 4(1 + \frac{1}{2\lambda^2})5}}{2(1 + \frac{1}{2\lambda^2})} \leq \frac{1}{3}(\sqrt{46} - 4)$$

□

The preceding lemma shows that our construction satisfies the Gap Invariant, and so we can now lower bound the distance growth during a single phase.

Lemma 2. *At the start of phase $i + 1$, we have $d_{i+1} \geq d_i \sqrt{1 + \frac{\alpha^2}{2\lambda^2}}$, where $\alpha = 0.927$ is an absolute constant.*

Proof. Suppose, without loss of generality, that the target is at q_a at the termination of the ith phase, which means the tracker is on or below the line $Q(t_i)q_c$. By the unit speed assumption, the target needs exactly $2d_i$ time for this move. The minimum value of d_{i+1} is at least as large as if P had moved directly to q_c by distance $2d_i$, as shown in Figure 2(ii). We can calculate d_{i+1} from the right triangle $q_a P(t_{i+1}) q_c$, as follows:

$$\begin{aligned}
d_{i+1} &\geq \sqrt{(\sqrt{4d_i^2 - \frac{\alpha^2 d_i^2}{\lambda^2}} - d_i)^2 + \frac{\alpha^2 d_i^2}{\lambda^2}} \\
&= \sqrt{d_i^2 - 2d_i \sqrt{4d_i^2 - \frac{\alpha^2 d_i^2}{\lambda^2}} + 4d_i^2 - \frac{\alpha^2 d_i^2}{\lambda^2} + \frac{\alpha^2 d_i^2}{\lambda^2}} \\
&= d_i \sqrt{5 - 4\sqrt{1 - \frac{\alpha^2}{4\lambda^2}}} \\
&\geq d_i \sqrt{5 - (4 - \frac{\alpha^2}{2\lambda^2})} \\
&= d_i \sqrt{1 + \frac{\alpha^2}{2\lambda^2}}
\end{aligned}$$

□

We can now prove the main result of this section.

Theorem 2. *Under the relative error localization model, a target can increase its distance from an equally fast tracker at the rate of $\Omega(\lambda^{-2})$. In other words, the target can ensure that $D(t) \geq D(0) + \Omega(t/\lambda^2)$ after any phase ending at time t.*

Proof. The target follows the phase strategy, where that after the ith phase that lasts $2d_i$ time units, the distance between the tracker and the target is at least $d_i \sqrt{1 + \frac{\alpha^2}{2\lambda^2}}$. Therefore, the distance increases during the ith phase by at least the following multiplicative factor (using a Taylor series expansion):

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$$\frac{d_i\sqrt{1 + \frac{\alpha^2}{2\lambda^2}} - d_i}{2d_i} \quad = \quad \frac{\sqrt{1 + \frac{\alpha^2}{2\lambda^2}} - 1}{2} \quad \geq \quad \frac{\alpha^2}{4\lambda^2} - \frac{\alpha^4}{16\lambda^4} \quad = \quad \Omega(\frac{1}{\lambda^2})$$

$\square$

3 Trackability with a Faster Tracker

The results of the previous section establish bounds on the relative advantage available to the target by the localization imprecision. Its distance from the tracker can grow at the rate of $\Theta(\lambda^{-2})$ with time. A tracking system can employ a number of different strategies to compensate for this disadvantage. In this section, we explore one such natural mechanism: *allow the tracker to move at a faster speed than the target.* A natural question then is: what is the minimum speedup necessary to cancel out the localization noise as a function of λ? We give bounds on the necessary and sufficient speedups, which match up to small constant factors as long as $\lambda \geq 2$. The general form of the speedup function is $(1 - \frac{1}{\lambda^2})^{-1/2}$. The following theorem proves the sufficiency condition.

Theorem 3. *Suppose the target moves with speed one, and the tracker has speed* $\sigma = \sqrt{\frac{1}{1-1/\lambda^2}}$*, where* λ *is the localization precision parameter. Then, the tracker can maintain* $D(t) \leq D(0)$*, for all times* $t \geq 0$*.*

Proof. Our analysis closely follows the proof of Theorem 1, and calculates the increase in the distance during time Δt. During this time, the tracker is able to move $\sigma \Delta t$, while the target can move at most Δt. We can then calculate distance at time $t + \Delta t$ from the triangle abc (Fig. 1), where $ab = \sigma \Delta t \sin\gamma$ and $ac = D(t) + \Delta t - \sigma \Delta t \cos\gamma$, as follows:

$$\begin{aligned}
D(t + \Delta t) &= \sqrt{(\sigma\Delta t \sin\gamma)^2 + (D(t) + \Delta t - \sigma\Delta t \cos\gamma)^2} \\
&= \sqrt{\sigma^2\Delta t^2 \sin(\alpha)^2 + D(t)^2 + 2D(t)\Delta t(1 - \sigma\cos(\alpha)) + \Delta t^2 - 2\Delta t^2\sigma\cos(\alpha) + \Delta t^2\sigma^2\cos(\alpha)^2} \\
&= \sqrt{\sigma^2\Delta t^2 + D(t)^2 + 2D(t)\Delta t(1 - \sigma\cos(\alpha)) + \Delta t^2 - 2\Delta t^2\sigma\cos(\alpha)} \\
&= \sqrt{(D(t) + \Delta t)^2 + \sigma^2\Delta t^2 - 2D(t)\Delta t\sigma\cos(\alpha) - 2\Delta t^2\sigma\cos(\alpha)} \\
&= \sqrt{(D(t) + \Delta t - \Delta t)^2 + \sigma^2\Delta t^2 + 2\Delta t(D(t) + \Delta t) - \Delta t^2 - 2D(t)\Delta t\sigma\cos(\alpha) - 2\Delta t^2\sigma\cos(\alpha)} \\
&= D(t)\sqrt{1 + \sigma^2\Delta t^2/D(t)^2 - \Delta t^2/D(t)^2 + 2\Delta t(D(t) + \Delta t)(1 - \sigma\cos(\alpha))/D(t)^2} \\
&\leq D(t) + \sigma^2\Delta t^2/2D(t) - \Delta t^2/2D(t) + \Delta t(1 + \Delta t/D(t))(1 - \sigma\cos\gamma)
\end{aligned}$$

This allows us to bound $D'(t) \leq 1 - \sigma\cos\gamma$, from which it follows that $D'(t) \leq 0$ as long as $\sigma \geq \sqrt{\frac{1}{1-1/\lambda^2}}$. $\square$

We now show that if $\lambda \geq 2$, this is the minimum speedup necessary as a function of λ, up to a small constant factor. We use the phase-based strategy of Theorem 2, however,

the value of α determined by Lemma 1 is not sufficient to maintain the Gap Invariant in this case because of the higher speed of the tracker. Instead, the following lemma gives the sufficient choice of α.

Lemma 3. *Let $\lambda \geq 2$ and and $\alpha \leq 0.68$ be a constant. Then, the Gap Invariant can be maintained in any phase as long as $\sigma \leq \frac{1}{\sqrt{1 - 1/\lambda^2}}$.*

Proof. As in Lemma 1 we need only show that $D(t) \geq \alpha d_i$, and again let d_{i+1} be the distance between P and Q if both moved toward q_a for the duration of phase i. Note that d_{i+1} is found as the length of the segment $P(t_i)q_a$ as in Lemma 1 except now minus $2\sigma d_i$ instead of $2d_i$. Thus, we have:

$$d_{i+1} = \sqrt{\left(\sqrt{4d_i^2 - \frac{d_i^2 \alpha^2}{\lambda^2}} + d_i\right)^2 + \frac{d_i^2 \alpha^2}{\lambda^2}} - 2\sigma d_i$$

$$= \sqrt{5d_i^2 - \frac{d_i^2 \alpha^2}{\lambda^2} + 2d_i\sqrt{4d_i^2 - \frac{d_i^2 \alpha^2}{\lambda^2}} + \frac{d_i^2 \alpha^2}{\lambda^2}} - 2\sigma d_i$$

$$= d_i\sqrt{5 + 4\sqrt{1 - \frac{\alpha^2}{4\lambda^2}}} - 2\sigma d_i$$

In order to satisfy the Gap Invariant, we must choose an α such that the following inequality holds:

$$\alpha d_i \leq d_i\sqrt{5 + 4\sqrt{1 - \frac{\alpha^2}{4\lambda^2}}} - 2\sigma d_i$$

$$\alpha^2 + 4\alpha\sigma + 4\sigma^2 \leq 5 + 4\sqrt{1 - \frac{\alpha^2}{4\lambda^2}}$$

$$\alpha^2 + 4\alpha\sigma + 4\sigma^2 \leq 5 + 4 - \frac{\alpha^2}{2\lambda^2}$$

$$\alpha^2(1 + \epsilon^2/2) + 4\alpha\sigma + 4\sigma^2 - 9 \leq 0$$

This gives the following upper bound when λ is minimum and σ is maximum, which by assumption is 2 and $1/\sqrt{1 - (1/2^2)}$, respectively.

$$\alpha \leq \frac{-4\sigma + \sqrt{16\sigma^2 - 4(1 + 1/2\lambda^2)(4\sigma^2 - 9)}}{2(1 + 1/2\lambda^2)} \leq 0.68$$

$\square$

We can now prove a lower bound on the increase in the distance during the ith phase.

Lemma 4. *If $\lambda \geq 2$, $\alpha \leq 0.68$, and $\sigma \leq (1 - 1/\lambda^2)^{-1/2}$, then at the start of the $i+1$ phase, we have $d_{i+1} \geq d_i\sqrt{(2\sigma - 3)^2 + \alpha^2(\sigma - 1/2)/\lambda^2}$, where $\alpha = 0.68$ is an absolute constant.*

Proof. Suppose, without loss of generality, that the target is at q_a at the termination of the ith phase, which means the tracker is below the line $Q(t_i)q_c$. By the unit speed assumption, the target needs exactly $2d_i$ time for this move. The minimum value of d_{i+1} is at least as large as if P had moved directly to q_c by distance $2\sigma d_i$, as shown in Figure 2(ii). We can calculate d_{i+1} from the right triangle $q_a P(t_{i+1})q_c$, as follows:

$$d_{i+1} \geq \sqrt{(\sqrt{4d_i^2 - \frac{\alpha^2 d_i^2}{\lambda^2}} - (2\sigma d_i - d_i))^2 + \frac{\alpha^2 d_i^2}{\lambda^2}}$$

$$= d_i \sqrt{5 + 4\sigma^2 - 4\sigma - 2(2\sigma - 1)\sqrt{4 - \frac{\alpha^2}{\lambda^2}}}$$

$$\geq d_i \sqrt{5 + 4\sigma^2 - 4\sigma - 4(2\sigma - 1)(1 - \alpha^2/8\lambda^2)}$$

$$= d_i \sqrt{4\sigma^2 - 12\sigma + 9 + \sigma\alpha^2/\lambda^2 - \alpha^2/2\lambda^2}$$

$$= d_i \sqrt{(2\sigma - 3)^2 + \alpha^2(\sigma - 1/2)/\lambda^2}$$

$\square$

Remark. The preceding lemma can be used to calculate the maximum tracker speed for which the target can still force a non-negative distance for a specific λ as follows:

$$\sqrt{(2\sigma - 3)^2 + \alpha^2(\sigma - 1/2)/\lambda^2} = 1$$

$$4\sigma^2 - 12\sigma + \frac{\alpha^2\sigma}{\lambda^2} = -8 + \frac{\alpha^2}{2\lambda^2}$$

$$2\sigma - \left(\frac{12 - (\alpha/\lambda)^2}{4}\right)^2 - \left(\frac{12 - \alpha^2/\lambda^2}{4}\right)^2 = -8 + \frac{\alpha^2}{2\lambda^2}$$

$$2\sigma - \frac{12 - \alpha^2/\lambda^2}{4} = -\sqrt{-8 + \frac{\alpha^2}{2\lambda^2} + (\frac{12 - \alpha^2/\lambda^2}{4})^2}$$

$$\sigma = \frac{-\sqrt{-8 + \frac{\alpha^2}{2\lambda^2} + (\frac{12-\alpha^2/\lambda^2}{4})^2} + \frac{12-\alpha^2/\lambda^2}{4}}{2}$$

As λ gets large, the upper and lower bound are within a constant factor of each other. Indeed, with a more careful choice of α, we can show that the upper and lower bounds are within a factor of 5.32 (as opposed to 10.23 for the above simple analysis) of each other for $\lambda \geq 2$, but we omit those details from this abstract.

4   Tracking in the Presence of Obstacles

The presence of obstacles makes the tracking problem considerably harder under the localization noise. The following simple example (Fig. 3) shows that the target can grow its

distance from the tracker as $D(t) \geq D(0) + t$, for any finite value of λ. The obstacle consists of a single U-shaped non-convex polygon. Initially, the target is at distance $D(0)$ from the tracker, and the "width" of the obstacle is less than $D(0)/2\lambda$, so that the localization error is unable to distinguish between a target moving inside the U channel, or around its outer boundary. One can show that no matter how the tracker pursues, its distance from the target can grow linearly with time.
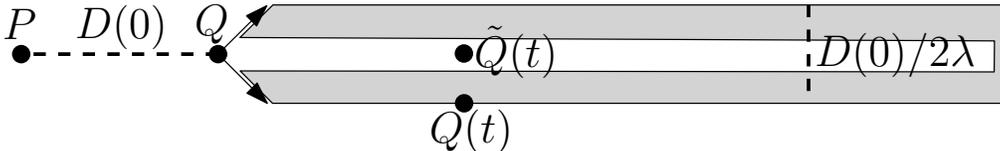

Figure 3: Impossibility of tracking among obstacles.

Path Proportionate Error. In order to get around this impossibility of tracking, we propose a *path proportionate error* measure, where the localization error is proportional to the *shortest path distance* between the target and the tracker, and not the Euclidean distance as used before. That is, the tracking signal and the physical movement of the agents follow the same path metric. Formally, the localization error at time t always obeys the following bound:

$$d(Q(t), \tilde{Q}(t)) \leq \frac{d(P(t), Q(t))}{\lambda}$$

We show that the best tracking performance in this model is $D(t) = D(0) + \Theta(t/\lambda)$; that is the distance grows linearly with $1/\lambda$, as opposed to the inverse quadratic function for the unobstructed case.

4.1 Tracking Upper Bound

The tracker's strategy in this case is also greedy, except now the tracker makes short-term commitments in *phases*, instead of continuously changing its path towards the new observed location. In particular, for each phase, the tracker fixes its goal as the *observed position of the target at the start of the phase*, moves along the shortest path to this goal, and then begins the next phase.

Algorithm 1 (MODIFIEDGREEDY). *The initial phase begins at time $t = 0$. During the ith phase, which begins at time t_i, the tracker moves along the shortest path to the observed location of the target at t_i, namely, $\tilde{Q}(t_i)$. When tracker reaches $\tilde{Q}(t_i)$, the ith phase ends, and the next phase begins.*

The upper bound on the tracking performance is given by the following theorem.

Theorem 4. *Using* MODIFIEDGREEDY, *the tracker can ensure that $D(t) \leq D(0) + O(t/\lambda)$.*

Proof. First note that because $d(\tilde{Q}(t_i), Q(t_i)) \leq D(t_i)/\lambda$, it follows that $t_{i+1} - t_i = D(t_i) + xD(t_i)$, where $\frac{-1}{\lambda} \leq x \leq \frac{1}{\lambda}$. Thus, the target's progress during the ith phase is upper bounded as $d(Q(t_i), Q(t_{i+1})) \leq D(t_i) + xD(t_i)$. Next, by applying the triangle inequality, the distance between P and Q at the beginning of phase t_{i+1} is upper bounded as

$$
\begin{aligned}
d(P(t_{i+1}), Q(t_{i+1})) &= d(\tilde{Q}(t_i), Q(t_{i+1})) \\
&\leq d(\tilde{Q}(t_i), Q(t_i)) + d(Q(t_i), Q(t_{i+1})) \\
&\leq \frac{D(t_i)}{\lambda} + D(t_i) + xD(t_i)
\end{aligned}
$$

Finally, the upper bound on the rate of distance increase can be derived as follows:

$$
\begin{aligned}
\frac{d(P(t_{i+1}), Q(t_{i+1})) - d(P(t_i), Q(t_i))}{t_{i+1} - t_i} &\leq \frac{D(t_i) + D(t_i)/\lambda + xD(t_i) - D(t_i)}{D(t_i) + xD(t_i)} \\
&= \frac{1/\lambda + x}{1 + x} \leq \frac{2}{\lambda + 1}
\end{aligned}
$$

where the final inequality uses the fact that the minimum value occurs when $x = 1/\lambda$. In conclusion, during each phase the distance between the tracker and the target increases by at most a factor of $\frac{2}{\lambda+1}$, giving the bound $D(t) \leq D(0) + O(\frac{t}{\lambda})$ ☐

4.2 Tracking Lower Bound.

Our final result is to prove that the trackability achieved by MODIFIEDGREEDY is essentially optimal. In particular, we construct an environment with polygonal obstacles and a movement strategy for the target that ensures $D(t) \geq D(0) + \Omega(t/\lambda)$. The construction of the polygonal environment is somewhat complicated and requires a carefully designed set of obstacles. The main schema of the construction is shown in Figure 4, where each edge of the "tree-like" diagram corresponds to a "channel" bounded by obstacles, and each face corresponds to a "gadget" consisting of a group of carefully constructed obstacles, with the outer face occupied entire by a single large obstacle.

As in the proof of Theorem 2, the target moves either to top or the bottom point of the gadget during a phase, depending on the tracker's location. The gadget construction is such that the movement of the target along either path is indistinguishable to the tracker because both paths are satisfied by a common set of observed locations throughout the path. Thus, by invoking the earlier equivalence principle, we may as well assume that the target knows the tracker's choices. If the target moves to the top, then the next phase occurs in the top gadget, otherwise the bottom, and so on.

To realize the geometric scheme of Figure 4, we replace each edge of the graph with a channel as shown in Figure 5(i). The desired edge length can be realized by adding any
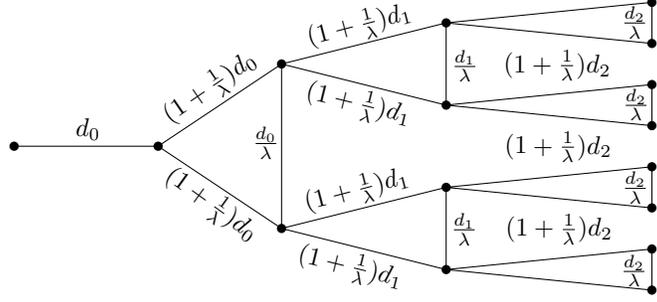
Figure 4: A high level schema for the lower bound construction. The numbers next to the edges denote the "path length" in the corresponding channels.

number of arbitrarily skinny bends such that the length of the shortest path through each channel equals the edge length. Each face is replaced with a set of obstacles, called a gadget, see Figure 5(ii) for an abstract illustration. The jagged line between each pair of nodes corresponds to a channel such that shortest path through that channel has the given length. The target will move along the shortest path through either the top or bottom channel while reporting its location in the center channel. Meanwhile, the channels connecting the top and bottom to the center will guarantee that $d(Q(t), \tilde{Q}(t)) \leq \frac{1}{\lambda} d(Q(t), P(t))$ at all times t during a phase.
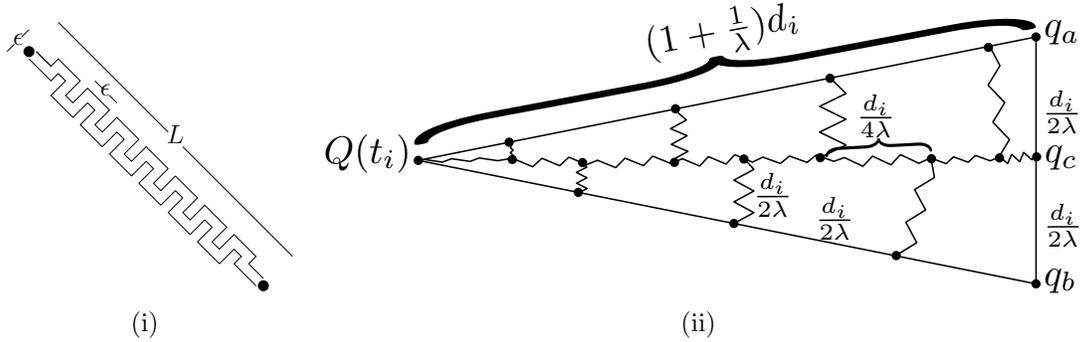

Figure 5: The channel construction in (i). In (ii) the shortest paths between nodes on the center path have length $\frac{d_i}{4\lambda}$, and the remaining all have length $\frac{d_i}{2\lambda}$.

4.2.1 Gadget Construction and its Properties

We now describe the construction of our gadgets and establish the geometric properties needed for the correctness of our lower bound. Each gadget is constructed out of two building blocks, the bent channels seen in Figure 5(i), and intersections depicted in Figure 6(i). Each intersection has the property that the shortest path between any two of the points among

a, b and c has length 2δ, where δ can be made arbitrarily close to 0. Thus we can construct a channel that branches into two channels such that the path length through the intersection is the same regardless of the branch chosen. In Figure 6(ii), we depict the construction of a gadget using only intersections (triangles) and channels (jagged lines).
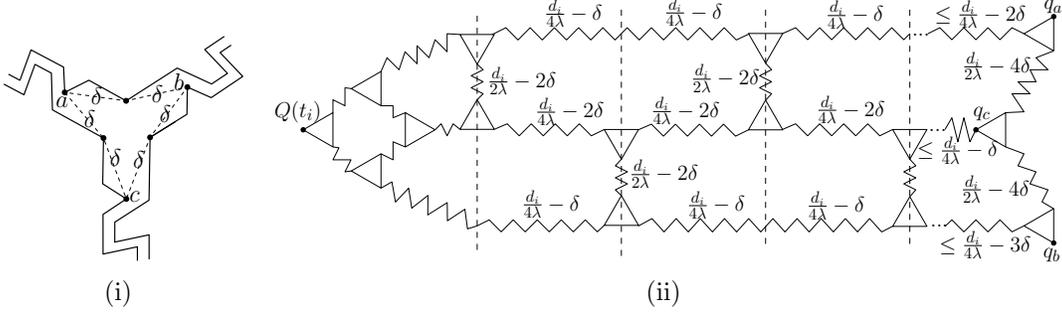

Figure 6: In (i) an example intersection such that the shortest path between any pair of a b and c has length 2δ. In (ii) an example gadget construction, where each triangle corresponds to an intersection with corners representing the points a b and c. The horizontal channels have length $\frac{d_i}{4\lambda}$ between each pair of vertical dashed lines, except for the initial distance before the first line (which can be made arbitrarily small), and the remaining spillover distance after the last dashed line.

As in the lower bound for the unobstructed case, the target starts the phase at $Q(t_i)$, and moves to q_a or q_b *while the observed location of the targets moves along the shortest path from $Q(t_i)$ to q_c.* In particular, let Π_a, Π_c, and Π_b denote the shortest paths from $Q(t_i)$ to q_a, q_c and q_b respectively. The following lemma establishes several properties needed for the feasibility of the target's strategy.

Lemma 5. *We can construct a gadget for each phase i such that (1) Π_a, Π_c and Π_b have length $(1 + \frac{1}{\lambda})d_i$ and (2) for any point x_c at distance ℓ along Π_c, the corresponding points x_a and x_b distance ℓ along Π_a and Π_b, respectively, satisfy $d(x_c, x_a) \leq \frac{d_i}{\lambda}$ and $d(x_c, x_b) \leq \frac{d_i}{\lambda}$.*

Proof. By construction, the shortest path in each channel between the dashed lines in Figure 6(ii) has length $\frac{d_i}{4\lambda}$, and therefore this construction can be extended until Π_a, Π_c and Π_b have length exactly $(1 + \frac{1}{\lambda})d_i$. Next, by the symmetry of the construction, we need only show that $d(x_c, x_a) \leq d_i/\lambda$. We ignore the case where x_c lies in the channels before the first dashed lines, as the length of such channels can be made arbitrarily small to guarantee that $d(x_a, x_c) \leq d_i/\lambda$. The maximum distance between x_a and x_c then occurs when x_a lies at the midpoint between two intersections in the top channel. However, in this case one can easily verify that the following holds:

$$d(x_c, x_a) = \delta + \frac{d_i}{4\lambda} - 2\delta + 2\delta + \frac{d_i}{2\lambda} - 2\delta + 2\delta + \frac{d_i}{4\lambda} - \delta = \frac{d_i}{\lambda}$$

This completes the proof. □

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4.2.2 Gap Invariant and the Proof of the Lower Bound

We now formulate the invariant maintained by the target so that its motion is valid under our (path proportionate) localization error and achieves the desired lower bound.

SP-Gap Invariant. Throughout the ith phase, the target moves along a path $Q(t)$ such that $D(t) \geq d_i$ for all times t, and all reported locations satisfy $d(Q(t), \tilde{Q}(t)) \leq \frac{d_i}{\lambda}$.

Lemma 6. *For the duration of phase i, SP-Gap Invariant is maintained.*

Proof. Whether Q moves along Π_a or Π_b, they are both shortest paths (and this cannot be shortcut by P), implying that $D(t) \geq d_i$ for the duration of the phase. Without loss of generality, suppose Q chooses Π_a. Then, after time t, both the target and its observed position have moved a distance of t along Π_a and Π_c, respectively. Therefore, by Lemma 5, we have $d(Q(t), \tilde{Q}(t)) \leq \frac{d_i}{\lambda}$. $\qquad\square$

We can prove our lower bound.

Theorem 5. *The target's strategy guarantees that after each phase ending at time t, the distance function satisfies $D(t) \geq D(0) + \Omega(\frac{t}{\lambda})$.*

Proof. The proof is by induction on the phase i. The basis of the induction is $i = 0$. Since the localization error makes target's top and bottom paths indistinguishable to the tracker, the target can ensure that at the end of phase 0 the target is on the side of Π_c that is opposite P. Without loss of generality, suppose that that target has reached q_a. Then the best case for P is if it moved $\frac{d_0}{\lambda}$ along Π_c, which achieves $D(t_1) \geq D(0) + \frac{D(0)}{2\lambda}$.

Now assume by induction that after phase $i-1$ ends at time t_i, we have $D(t_i) \geq D(t_{i-1}) + D(t_{i-1})/2\lambda = d_i$. Suppose now that P has yet to reach the gadget corresponding to phase i when Q has finished phase i at time t_{i+1}. Then necessarily $D(t_{i+1}) \geq d_i + d_i/\lambda$, as that is the length Π_a and Π_b. Otherwise if P has moved into the gadget, then the inequality $D(t_i) \geq d_i$ ensures that the closest the target can be to the tracker is if P has moved $\frac{d_i}{\lambda}$ along Π_c, which implies $D(t_{i+1}) \geq D(t_i) + \frac{D(t_i)}{2\lambda}$.

Thus, in a round with duration $(1 + \frac{1}{\lambda})d_i$, the distance increases by at least $d_i/2\lambda$. Thus, in the ith phase, the distance increases by a factor of at least

$$\frac{d_i/2\lambda}{(1 + \frac{1}{\lambda})d_i} = \frac{1}{2\lambda(1 + \frac{1}{\lambda})} = \frac{1}{2 + 2\lambda}$$

Thus, at the end of any phase, we have the inequality $D(t) \geq D(0) + \Omega(t/\lambda)$, which completes the lower bound. $\qquad\square$

5 Extension to d dimensions

Our analysis of trackability was carried out for 2-dimensional Euclidean plane, but the results generalize easily to d dimensions. Indeed, in the unobstructed case, our analysis of the upper

bound only makes use of the triangle inequality: the region of interest is the triangle formed by $P(t)$, $Q(t)$, and $\tilde{Q}(t)$, and the target Q moves directly away from P. Thus, within an arbitrarily small time interval Δt, P and Q are moving within the two-dimensional plane of the triangle $P(t)Q(t)\tilde{Q}(t)$. The upper bound analysis therefore extend to any dimension $d \geq 2$. The same reasoning also holds in the presence of obstacles. Finally, the lower bound construction of $d = 2$ immediately implies that the trackability lower bound holds in all dimensions $d \geq 2$.

6   Conclusion

Our paper is an attempt to formally study the worst-case impact of noisy localization on the performance of tracking systems. We analyzed a simple, but fundamental, problem where a tracker wants to pursue a target but the tracker's location sensor can measure the target's position only approximately, with a relative error parameterized by quantity λ. We prove upper and lower bound on the tracking performance as a function of this localization parameter λ. A few surprising consequences of our results are (1) that the naive strategy of "always move to the observed location" is asymptotically optimal, (2) the tracking error has a nice analytic form, showing an inverse quadratic dependence on λ, and (3) under the natural "path proportional error" for environments with obstacles, the trackability has qualitatively a different dependence of the form $\Omega(1/\lambda)$.

Compared to often-used heuristics such as Kalman filters, our work offers a more theoretical perspective for analyzing motion and localization errors in the presence of inevitable noise, which may be especially useful in situations where worst-case bounds are important, such as adversarial tracking or surveillance. In addition to improving the constant factors in our bounds, it will also be interesting to study the noisy sensing model for other more complex settings.

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