Controllability Analysis for Multirotor Helicopter Rotor Degradation and Failure

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NOMENCLATURE

h	=	altitude of the helicopter, m
ϕ, θ, ψ	=	roll, pitch and yaw angles of the helicopter, rad
v_h	=	vertical velocity of the helicopter, m/s
p, q, r	=	roll, pitch and yaw angular velocities of the helicopter, rad/s
T	=	total thrust of the helicopter, N
L, M, N	=	airframe roll, pitch and yaw torque of the helicopter, N·m
m_a	=	mass of the helicopter, kg
g	=	acceleration of gravity, kg·m/s^2
J_x, J_y, J_z	=	moment of inertia around the roll, pitch and yaw axes of the helicopter frame, kg·m^2
f_i	=	lift of the i-th rotor, N
K_i	=	maximum lift of the i-th rotor, N
η_i	=	efficiency parameter of the i-th rotor
r_i	=	distance from the center of the i-th rotor to the center of mass, m
m	=	number of rotors
k_μ	=	ratio between the reactive torque and the lift of the rotors

I. INTRODUCTION

Multirotor helicopters [1], [2], [3] are attracting increasing attention in recent years because of their important contribution and cost effective application in several tasks such as surveillance, search and

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rescue missions and so on. However, there exists a potential risk to civil safety if a mutirotor aircraft crashes, especially in an urban area. Therefore, it is of great importance to consider the flight safety of multirotor helicopters in the presence of rotor faults or failures [4].

Fault-Tolerant Control (FTC) [5] has the potential to improve the safety and reliability of multirotor helicopters. FTC is the ability of a controlled system to maintain or gracefully degrade control objectives despite the occurrence of a fault [6]. There are many applications in which fault tolerance may be achieved by using adaptive control, reliable control**,** or reconfigurable control strategies [7], [8]. Some strategies involve explicit fault diagnosis, and some do not. The reader is referred to a recent survey paper [9] for an outline of the state of art in the field of FTC. However, only few attempts are known that focus on the fundamental FTC property analysis, one of which is defined as the (control) reconfigurability [6]. A faulty multirotor system with inadequate reconfigurability cannot be made to effectively tolerate faults regardless of the feedback control strategy used [10]. The control reconfigurability can be analyzed from the intrinsic and performance-based perspectives. The aim of this Note is to analyze the control reconfigurability for multirotor systems (4-, 6- and 8-rotor helicopters, etc.) from the controllability analysis point of view**.**

Classical controllability theories of linear systems are not sufficient to test the controllability of the considered multirotor helicopters, as the rotors can only provide unidirectional lift (upward or downward) in practice. In our previous work [11], it was shown that a hexacopter with the standard symmetrical configuration is uncontrollable if one rotor fails, though the controllability matrix of the hexacopter is row full rank. Thus, the reconfigurability based on the controllability Gramian [10] is no longer applicable. Brammer in [12] proposed a necessary and sufficient condition for the controllability of linear autonomous systems with positive constraint, which can be used to analyze the controllability of multirotor systems. However, the theorems in [12] are not easy to use in practice. Owing to this, the controllability of a given system is reduced to those of its subsystems with real eigenvalues based on the Jordan canonical form in [13]. However, appropriate stable algorithms to compute Jordan real canonical form should be used to avoid ill-conditioned calculations. Moreover, a step-by-step controllability test procedure is not given. To address these problems, in this Note the theory proposed in [12] is extended and a new necessary and sufficient condition of controllability is derived for the considered multirotor systems.

Nowadays, larger multirotor aircraft are starting to emerge and some multirotor aircraft are controlled by varying the collective pitch of the blade. This work considers only the multirotor helicopters controlled by varying the RPM (Revolutions Per Minute) of each rotor but this research can be extended to most multirotor aircraft regardless of size whether they are controlled by varying the collective pitch of the blade or the RPM.

The linear dynamical model of the considered multirotor helicopters around hover conditions is derived first, and then the control constraint is specified. It is pointed out that classical controllability theories of linear systems are not sufficient to test the controllability of the derived model (Section II). Then the controllability of the derived model is studied based on the theory in [12], and two conditions which are necessary and sufficient for the controllability of the derived model are given. In order to make the two conditions easy to test in practice, an Available Control Authority Index (ACAI) is introduced to quantify the available control authority of the considered multirotor systems. Based on the ACAI, a new necessary and sufficient condition is given to test the controllability of the considered multirotor systems (Section III). Furthermore, the computation of the proposed ACAI and a step-by-step controllability test procedure is approached for practical application (Section IV). The proposed controllability test method is used to analyze the controllability of a class of hexacopters to show its effectiveness (Section V). The major contributions of this Note are: (i) an ACAI to quantify the available control authority of the considered multirotor systems, (ii) a new necessary and sufficient controllability test condition based on the proposed ACAI, and (iii) a step-by-step controllability test procedure for the considered multirotor systems.

II. Problem Formulation

This Note considers a class of multirotor helicopters shown in Fig.1, which are often used in practice. From Fig.1, it can be seen that there are various types of multirotor helicopters with different

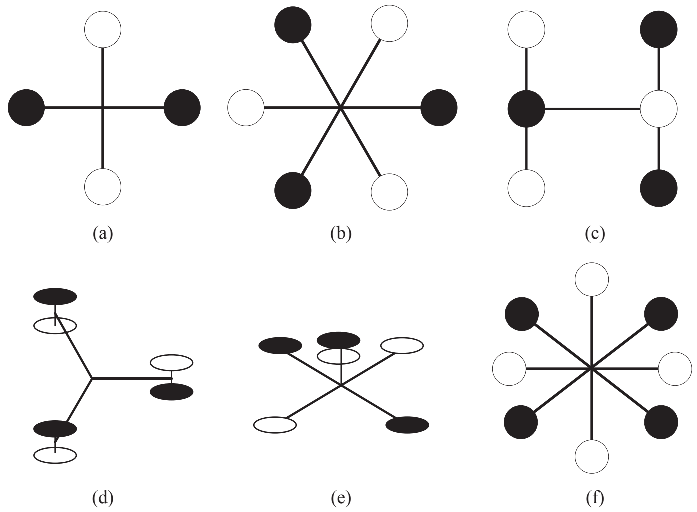

Fig. 1. Different configurations of multirotor helicopters (the white disc denotes that the rotor rotates clockwise and the black disc denotes that the rotor rotates anticlockwise)

rotor numbers and different configurations. Despite the difference in type and configuration, they can all be modeled in a general form as equation (1). In reality, the dynamical model of the multirotor helicopters is nonlinear and there are some aerodynamic damping and stiffness. But if the multirotor helicopter is hovering, the aerodynamic damping and stiffness is ignorable. The linear dynamical model around hover conditions is given as [14], [15], [16]:

$$\dot{x} = Ax + B\underbrace{(F - G)}_{u} \tag{1}$$

where

$$x = [h\ \phi\ \theta\ \psi\ v_h\ p\ q\ r]^T \in \mathbb{R}^8, F = [T\ L\ M\ N]^T \in \mathbb{R}^4, G = [m_a g\ 0\ 0\ 0]^T \in \mathbb{R}^4,$$

$$A = \begin{bmatrix} 0_{4\times 4} & I_4 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{8\times 8}, B = \begin{bmatrix} 0 \\ J_f^{-1} \end{bmatrix} \in \mathbb{R}^{8\times 4}, J_f = \text{diag}\left(-m_a, J_x, J_y, J_z\right)$$

In practice, $f_i \in [0, K_i], i = 1, \cdots m$ since the rotors can only provide unidirectional lift (upward or downward). As a result, the rotor lift f is constrained by

$$f \in \mathcal{F} = \Pi_{i=1}^{m} [0, K_i]. \tag{2}$$

Then according to the geometry of the multirotor system shown in Fig.2, the mapping from the rotor

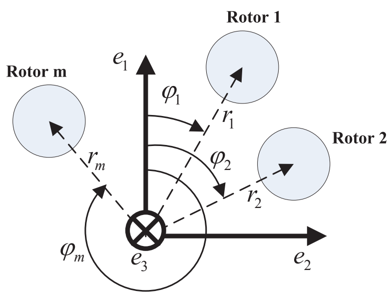

Fig. 2. Geometry definition for multirotor system

lift $f_i, i = 1, \cdots m$ to the system total thrust/torque F is:

$$F = B_f f \tag{3}$$

where $f = [f_1 \ \cdots \ f_m]^T$. The matrix $B_f \in \mathbb{R}^{4\times m}$ is the control effectiveness matrix and

$$B_f = [b_1 \ b_2 \ \cdots \ b_m] \tag{4}$$

where $b_i = \eta_i \bar{b}_i$, $\bar{b}_i \in \mathbb{R}^4, i \in \{1, \cdots m\}$ is the vector of contribution factors of the i-th rotor to the total thrust/torque F, the parameters $\eta_i \in [0, 1], i = 1, \cdots, 6$ is used to account for rotor wear/failure. If the i-th rotor fails, then $\eta_i = 0$. For a multirotor helicopter whose geometry is shown in Fig.2, the control effectiveness matrix B_f in parameterized form is [16]

$$B_f = \begin{bmatrix} \eta_1 & \cdots & \eta_m \\ -\eta_1 r_1 \sin(\varphi_1) & \cdots & -\eta_m r_m \sin(\varphi_m) \\ \eta_1 r_1 \cos(\varphi_1) & \cdots & \eta_m r_m \cos(\varphi_m) \\ \eta_1 w_1 k_\mu & \cdots & \eta_m w_m k_\mu \end{bmatrix} \tag{5}$$

where w_i is defined by

$$w_i = \begin{cases} 1, \text{ if rotor } i \text{ rotates anticlockwise} \\ -1, \text{ if rotor } i \text{ rotates clockwise} \end{cases}. \tag{6}$$

By (2) and (3), F is constrained by

$$\Omega = \{F | F = B_f f, f \in \mathcal{F}\}. \tag{7}$$

Then u is constrained by

$$\mathcal{U} = \{u | u = F - G, F \in \Omega\}. \tag{8}$$

From (2) (7) and (8), $\mathcal{F}, \Omega, \mathcal{U}$, are all convex and closed.

Our major objective is to study the controllability of the system (1) under the constraint $\mathcal{U}$.

Remark 1. The system (1) with constraint set $\mathcal{U} \subset \mathbb{R}^4$ is called controllable if, for each pair of points $x_0 \in \mathbb{R}^8$ and $x_1 \in \mathbb{R}^8$, there exists a bounded admissible control, $u(t) \in \mathcal{U}$, defined on some finite interval $0 \le t \le t_1$, which steers x_0 to x_1. Specifically, the solution to (1), $x(t, u(\cdot))$, satisfies the boundary conditions $x(0, u(\cdot)) = x_0$ and $x(t_1, u(\cdot)) = x_1$.

Remark 2. Classical controllability theories of linear systems often require the origin to be an interior point of $\mathcal{U}$ so that $\mathcal{C}(A, B)$ being row full rank is a necessary and sufficient condition [12]. However, the origin is not always inside control constraint $\mathcal{U}$ of the system (1) under rotor failures. Consequently, $\mathcal{C}(A, B)$ being row full rank is not sufficient to test the controllability of the system (1).

III. Controllability for the Multirotor Systems

In this section, the controllability of the system (1) is studied based on the positive controllability theory proposed in [12]. Applying the positive controllability theorem in [12] to the system (1) directly, the following theorem is obtained

Theorem 1. The following conditions are necessary and sufficient for the controllability of the system (1):

(i) Rank $\mathcal{C}(A, B) = 8$, where $\mathcal{C}(A, B) = \begin{bmatrix} B & AB & \cdots & A^7 B \end{bmatrix}$.

(ii) There is no real eigenvector v of A^T satisfying $v^T Bu \le 0$ for all $u \in \mathcal{U}$.

It is difficult to test the condition (ii) in *Theorem 1*, because in practice one cannot check all u in $\mathcal{U}$. In the following, an easy-to-use criterion is proposed to test the condition (ii) in *Theorem 1*.

Before going further, a measure is defined as:

$$\rho\left(X,\partial\Omega\right) \triangleq \begin{cases} \min\left\{\|X-F\| : X\in\Omega, F\in\partial\Omega\right\} \\ -\min\left\{\|X-F\| : X\in\Omega^{C}, F\in\partial\Omega\right\} \end{cases} \tag{9}$$

where $\partial\Omega$ is the boundary of Ω and Ω^C is the complementary set of Ω. If $\rho\left(X,\partial\Omega\right)\leq 0$, then $X\in\Omega^C\cup\partial\Omega$, which means that X is not an interior point of Ω. Otherwise, X is an interior point of Ω.

According to (9), $\rho\left(G,\partial\Omega\right)=\min\left\{\|G-F\|, F\in\partial\Omega\right\}$ which is the radius of the biggest enclosed sphere centered at G in the attainable control set Ω. In practice, it is the maximum control thrust/torque that can be produced in all directions. Therefore, it is an important quantity to ensure controllability for arbitrary rotor wear/failure. Then $\rho\left(G,\partial\Omega\right)$ can be used to quantify the available control authority of the system (1). From (8), it can be seen that all the elements in $\mathcal{U}$ are given by translating the all the elements in Ω by a constant G. As translation does not change the relative position of all the elements of Ω, the value of $\rho\left(0,\partial\mathcal{U}\right)$ is equal to the value of $\rho\left(G,\partial\Omega\right)$. In this Note, the Available Control Authority Index (ACAI) of system (1) is defined by $\rho\left(G,\partial\Omega\right)$ as Ω is the attainable control set and more intuitive than $\mathcal{U}$ in practice. The ACAI shows the ability as well as the control capacity of a multirotor helicopter controlling its altitude and attitude. With this definition, the following lemma about condition (ii) of *Theorem 1* is obtained.

Lemma 1: The following three statements are equivalent for the system (1):

(i) There is no non-zero real eigenvector v of A^T satisfying $v^T Bu \leq 0$ for all $u\in\mathcal{U}$ or $v^T B\left(F-G\right)\leq 0$ for all $F\in\Omega$.

(ii) G is an interior point of Ω.

(iii) $\rho\left(G,\partial\Omega\right)>0$.

Proof: See *Appendix A.* □

By *Lemma 1,* condition (ii) in *Theorem 1* can be tested by the value $\rho\left(G,\partial\Omega\right)$. Now a new necessary and sufficient condition can be derived to test the controllability of the system (1).

Theorem 2: System (1) is controllable, if and only if the following two conditions hold:

(i) Rank $\mathcal{C}(A,B)=8$.

(ii) $\rho(G,\partial\Omega)>0$.

According to *Lemma 1*, *Theorem 2* is straightforward from *Theorem 1*. Actually, *Theorem 2* is a corollary of *Theorem 1.4* presented in [12]. To make this Note more readable and self-contained, we extend the condition (1.6) of *Theorem 1.4* presented in [12], and get the condition (ii) in *Theorem 2* of this Note based on the simplified structure of (A,B) pair and the convexity of $\mathcal{U}$. This extension can enable the quantification of the controllability and also make it possible to develop a step-by-step controllability test procedure for the multirotor systems. In the following section, a step-by-step controllability test procedure is approached based on *Theorem 2*.

IV. A Step-by-Step Controllability Test Procedure

This section will show how to obtain the value of the proposed ACAI in Section III. Furthermore, a step-by-step controllability test procedure for the controllability of the system (1) is approached for practical applications.

A. Available Control Authority Index Computation

First, two index matrices S_1 and S_2 are defined, where S_1 is a matrix whose rows consist of all possible combinations of 3 elements of $M=[1\ 2\ \cdots\ m]$, and the corresponding rows of S_2 are the remaining $m-3$ elements of M. The matrix S_1 contains s_m rows and 3 columns, and the matrix S_2 contains s_m rows and $m-3$ columns, where

$$s_m=\frac{m!}{(m-(n_\Omega-1))!\,(n_\Omega-1)!}. \tag{10}$$

For the system in equation (1), s_m is the number of the groups of parallel boundary segments in $\mathcal{F}$. For example, if $m=4$, $n_\Omega=4$, then $s_m=4$ and

$$S_1=\begin{bmatrix}1&2&3\\1&2&4\\1&3&4\\2&3&4\end{bmatrix},S_2=\begin{bmatrix}4\\3\\2\\1\end{bmatrix}$$

Define $B_{1,j}$ and $B_{2,j}$ as follows:

$$B_{1,j} = [b_{S_1(j,1)}\ b_{S_1(j,2)}\ b_{S_1(j,3)}] \in \mathbb{R}^{4\times 3}$$
$$B_{2,j} = [b_{S_2(j,1)}\ \cdots\ b_{S_2(j,m-3)}] \in \mathbb{R}^{4\times (m-3)} \tag{11}$$

where $j = 1, \cdots, s_m$, $S_1(j, k_1)$ is the element at the j-th row and the k_1-th column of S_1, and $S_2(j, k_2)$ is the element at the j-th row and the k_2-th column of S_2. Here $k_1 = 1, 2, 3$ and $k_2 = 1, \cdots, m-3$.

Define a sign function sign$(\cdot)$ as follows: for an n dimensional vector $a = [a_1\ \cdots\ a_n] \in \mathbb{R}^{1\times n}$,

$$\text{sign}(a) = [c_1\ \cdots\ c_n] \tag{12}$$

where $c_i = 1$ if $a_i > 0$, $c_i = 0$ if $a_i = 0$, and $c_i = -1$ if $a_i < 0$. Then $\rho(G, \partial\Omega)$ is obtained by the following theorem.

Theorem 3. For the system in equation (1), if rank $B_f = 4$ then the ACAI $\rho(G, \partial\Omega)$ is given by

$$\rho(G, \partial\Omega) = \text{sign}(\min(d_1, d_2, \cdots, d_{s_m})) \min(|d_1|, |d_2|, \cdots, |d_{s_m}|). \tag{13}$$

If rank $B_{1,j} = 3$, then

$$d_j = \frac{1}{2}\text{sign}\left(\xi_j^T B_{2,j}\right) \Lambda_j \left(\xi_j^T B_{2,j}\right)^T - \left|\xi_j^T (B_f f_c - G)\right|, j = 1, \cdots, s_m \tag{14}$$

where $f_c = \frac{1}{2}[K_1\ K_2\ \cdots K_m]^T \in \mathbb{R}^m$ and $\Lambda_j \in \mathbb{R}^{(m-3)\times(m-3)}$ is given by

$$\Lambda_j = \begin{bmatrix} K_{S_2(j,1)} & 0 & 0 & 0 \\ 0 & K_{S_2(j,2)} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & K_{S_2(j,m-3)} \end{bmatrix} \tag{15}$$

The vector $\xi_j \in \mathbb{R}^4$ satisfies

$$\xi_j^T B_{1,j} = 0, \|\xi_j\| = 1 \tag{16}$$

and $B_{1,j}$ and $B_{2,j}$ are given by (11). If rank $B_{1,j} < 3$, $d_j = +\infty$.

Proof: The proof process is divided into 3 steps and the details can be found in *Appendix B*. □

Remark 3. In practice, $+\infty$ is replaced by a sufficiently large positive number (for example, set $d_j = 10^6$). If rank $B_f < 4$, then Ω is not a 4 dimensional hypercube and the ACAI makes no sense which is set to $-\infty$. Similarly, $-\infty$ is replaced by -10^6 in practice). From (13), if $\rho\left(G, \partial\Omega\right) > 0$, then G is an interior point of Ω and $\rho\left(G, \partial\Omega\right)$ is the minimum distance from G to $\partial\Omega$. If $\rho\left(G, \partial\Omega\right) < 0$, then G is not an interior point of Ω and $\left|\rho\left(G, \partial\Omega\right)\right|$ is the minimum distance from G to $\partial\Omega$. The ACAI $\rho\left(G, \partial\Omega\right)$ can also be used to show a degree of controllability (see [17], [18], [19]) of the system in equation (1), but the ACAI is fundamentally different from the degree of controllability in [17]. The degree of controllability in [17] is defined based on the minimum Euclidean norm of the state on the boundary of the recovery region for time t. However, the ACAI is defined based on the minimum Euclidean norm of the control force on the boundary of the attainable control set. The degree of controllability in [17] is time-dependent, whereas the ACAI is time-independent. A very similar multirotor failure assessment was provided in [16] by computing the radius of the biggest circle that fits in the L-M plane with the center in the origin ($L = 0$, $M = 0$), where the L-M plane is obtained by cuting the four-dimensional attainable control set at the nominal hovering conditions defined with $T = G$ and $N = 0$. This computation is very simple and intuitive. But the radius of the two-dimensional L-M plane can only quantify the control authority of roll and pitch control. To account for this, the ACAI proposed by this Note is defined by the radius of the biggest ball that fits in the four-dimensional polytopes Ω with the center in G.

B. Controllability Test Procedure for Multirotor Systems

From the above, the controllability of the multirotor system (1) can be analyzed by the following procedure:

Step 1: Check the rank of $\mathcal{C}\left(A, B\right)$. If $\mathcal{C}\left(A, B\right) = 8$, go to *Step 2*. If $\mathcal{C}\left(A, B\right) < 8$, go to *Step 9*.

Step 2: Set the value of the rotor's efficiency parameter η_i,$i = 1, \cdots, m$ to get $B_f = \left[b_1 \; b_2 \; \cdots \; b_m\right]$ as shown in (4). If rank $B_f = 4$, go to *Step 3*. If rank $B_f < 4$, let $\rho\left(G, \partial\Omega\right) = -10^6$ and go to *Step 9*.

Step 3: Compute the two index matrices S_1 and S_2, where S_1 is a matrix whose rows consist of

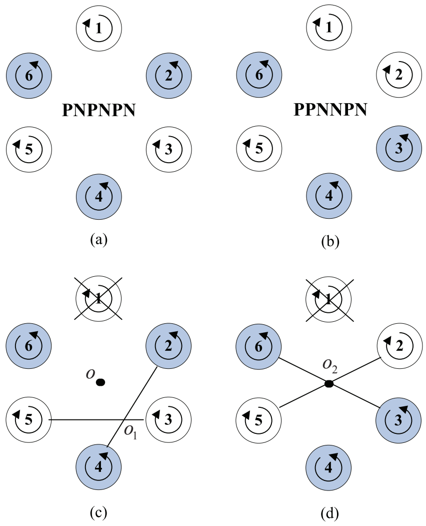

Fig. 3. (a) Standard rotor arrangement, (b) new rotor arrangement, (c) the 1-st rotor of the PNPNPN system fails, (d) the 1-st rotor of the PPNNPN system fails.

all possible combinations of the m elements of M taken 3 at a time and the rows of S_2 are the remaining $(m-3)$ elements of M, $M = [1\ 2\ \cdots\ m]$.

Step 4: $j = 1$.

Step 5: Compute the two matrices $B_{1,j}$ and $B_{2,j}$ according to (11).

Step 6: If rank $B_{1,j} = 3$, compute d_j according to (14). If rank $B_{1,j} < 3$, set $d_j = 10^6$.

Step 7: $j = j + 1$. If $j \leq s_m$, go to *Step 5*. If $j > s_m$, go to *Step 8*.

Step 8: Compute $\rho\left(G, \partial\Omega\right)$ according to (13).

Step 9: If $\mathcal{C}\left(A, B\right) < 8$ or $\rho\left(G, \partial\Omega\right) \leq 0$, the system (1) is uncontrollable. Otherwise, the system in equation (1) is controllable.

V. Controllability Analysis for a Class of Hexacopters

In this section, the controllability test procedure developed in section IV is used to analyze the controllability of a class of hexacopters shown in Fig.3, subject to rotor wear/failures, to show its effectiveness.

The rotor arrangement of the considered hexacopter is the standard symmetrical configuration shown in Fig.3(a). PNPNPN is used to denote the standard arrangement, where “P” denotes that

TABLE I

HEXACOPTER PARAMETERS

Parameter	Value	Units
m_a	1.535	kg
g	9.80	m/s^2
$r_i, i = 1, \cdots, 6$	0.275	m
$K_i, i = 1, \cdots, 6$	6.125	N
J_x	0.0411	kg·m^2
J_y	0.0478	kg·m^2
J_z	0.0599	kg·m^2
k_μ	0.1	-

TABLE II

HEXACOPTER (PNPNPN) CONTROLLABILITY WITH ONE ROTOR FAILED

Rotor failure	Rank of $\mathcal{C}(A, B)$	ACAI	Controllability
No wear/failure	8	1.4861	controllable
$\eta_1 = 0$	8	0	uncontrollable
$\eta_2 = 0$	8	0	uncontrollable
$\eta_3 = 0$	8	0	uncontrollable
$\eta_4 = 0$	8	0	uncontrollable
$\eta_5 = 0$	8	0	uncontrollable
$\eta_6 = 0$	8	0	uncontrollable

a rotor rotates clockwise and “N” denotes that a rotor rotates anticlockwise. According to (4), the control effectiveness matrix B_f of that hexacopter configuration is

$$B_f = \begin{bmatrix} \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 & \eta_6 \\ 0 & -\frac{\sqrt{3}}{2}\eta_2 r_2 & -\frac{\sqrt{3}}{2}\eta_3 r_3 & 0 & \frac{\sqrt{3}}{2}\eta_5 r_5 & \frac{\sqrt{3}}{2}\eta_6 r_6 \\ \eta_1 r_1 & \frac{1}{2}\eta_2 r_2 & -\frac{1}{2}\eta_3 r_3 & -\eta_4 r_4 & -\frac{1}{2}\eta_5 r_5 & \frac{1}{2}\eta_6 r_6 \\ -\eta_1 k_\mu & \eta_2 k_\mu & -\eta_3 k_\mu & \eta_4 k_\mu & -\eta_5 k_\mu & \eta_6 k_\mu \end{bmatrix} \tag{17}$$

Using the procedure defined in Section IV, the controllability analysis results of the PNPNPN hexacopter subject to one rotor failure is shown in Table II. The PNPNPN hexacopter is uncontrollable when one rotor fails, even though its controllability matrix is row full rank. A new rotor arrangement (PPNNPN) of the hexacopter shown in Fig.3(b) is proposed in [16], which is still controllable when one of some specific rotors stops. The controllability of the PPNNPN hexacopter subject to one rotor failure is shown in Table III.

TABLE III

HEXACOPTER (PPNNPN) CONTROLLABILITY WITH ONE ROTOR FAILED

Rotor failure	Rank of $\mathcal{C}(A, B)$	ACAI	Controllability
No wear/failure	8	1.1295	controllable
$\eta_1 = 0$	8	0.7221	controllable
$\eta_2 = 0$	8	0.4510	controllable
$\eta_3 = 0$	8	0.4510	controllable
$\eta_4 = 0$	8	0.7221	controllable
$\eta_5 = 0$	8	0	uncontrollable
$\eta_6 = 0$	8	0	uncontrollable

From Table II and Table III, the value of the ACAI is 1.4861 for the PNPNPN hexacopter subject to no rotor failures, while the value of the ACAI is reduced to 1.1295 for the PPNNPN hexacopter. It can be observed that the use of the PPNNPN configuration instead of the PNPNPN configuration improves the fault-tolerance capabilities but also decreases the ACAI for the no failure condition. Similar to the results in [16], changing the rotor arrangement is always a tradeoff between fault-tolerance and control authority. That said, the PPNNPN system is not always controllable under a failure. From Table III, it can be seen that if the 5-th rotor or the 6-th rotor fails the PPNNPN system is uncontrollable.

The following provides some physical insight between the two configurations. For the PPNNPN configuration, if one of the rotors (other than the 5-th and 6-th rotor) of that system fails, the remaining rotors still comprise a basic quadrotor configuration that is symmetric about the mass center (see Fig.3(d)). In contrast, if one rotor of the PNPNPN system fails, although the remaining rotors can

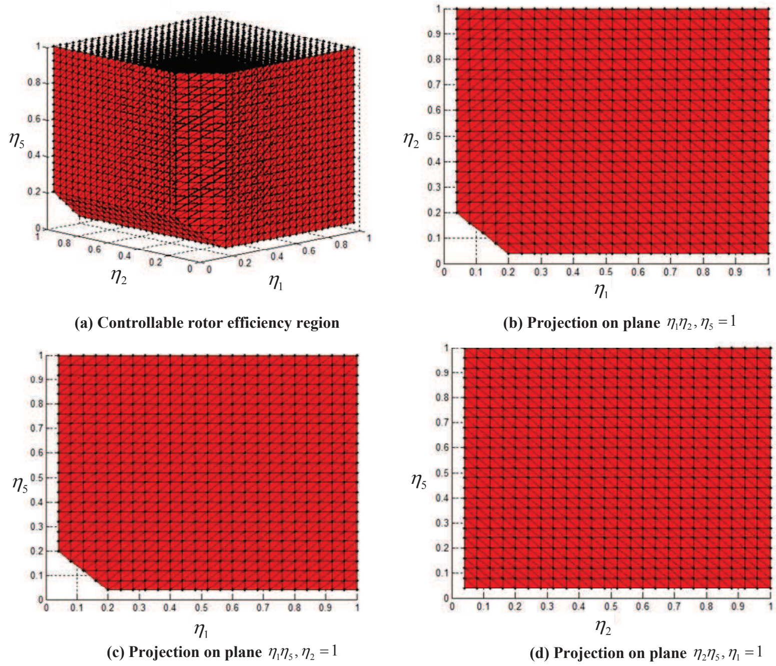

Fig. 4. Controllable region of different rotors' efficiency parameter for the PNPNPN hexacopter

make up a basic quadrotor configuration, the quadrotor configuration is not symmetric about the mass center (see Fig.3(c)). The result is that the PPNNPN system under most single rotor failures can provide the necessary thrust and torque control, while the PNPNPN system cannot.

Therefore, it is necessary to test the controllability of the multirotor helicopters before any fault-tolerant control strategies are employed. Moreover, the controllability test procedure approached can also be used to test the controllability of the hexacopter with different η_i, $i \in \{1, \cdots, 6\}$. Let η_1, η_2, η_5 vary in $[0,1] \subset \mathbb{R}$, namely rotor 1, rotor 2 and rotor 5 are worn; then the PNPNPN hexacopter retains controllability while η_1, η_2, η_5 are in the grid region (where the grid spacing is 0.04) in Fig.4. The corresponding ACAI at the boundaries of the projections shown in Fig. 4 is zero or near to zero (because of error in numerical calculation).

VI. Conclusions

The controllability problem of a class of multirotor helicopters was investigated. An Available Control Authority Index (ACAI) was introduced to quantify the available control authority of multirotor systems. Based on the ACAI, a new necessary and sufficient condition was given based on a positive controllability theory. Moreover, a step-by-step procedure was developed to test the controllability of the considered multirotor helicopters. The proposed controllability test method was used to analyze the controllability of a class of hexacopters to show its effectiveness. Analysis results showed that the hexacopters with different rotor configurations have different fault tolerant capabilities. It is therefore necessary to test the controllability of the multirotor helicopters before any fault-tolerant control strategies are employed.

APPENDIX

A. Proof of Lemma 1

In order to make this Note self-contained, the following lemma is introduced:

Lemma 3 [20]. If Ω is a nonempty convex set in $\mathbb{R}^4$ and F_0 is not an interior point of Ω, then there is a nonzero vector k such that $k^T(F - F_0) \leq 0$ for each $F \in cl(\Omega)$, where $cl(\Omega)$ is the closure of Ω.

Then according to *Lemma 3*,

(i)⇒(ii): Suppose that (i) holds. It is easy to see that all the eigenvalues of A^T are zero. By solving the linear equation $A^T v = 0$, all the eigenvectors of A^T are expressed in the following form

$$v = [0\ 0\ 0\ 0\ k_1\ \ k_2\ \ k_3\ \ k_4]^T \tag{18}$$

where $v \neq 0, k = [k_1\ \ k_2\ \ k_3\ \ k_4]^T \in \mathbb{R}^4$, and $k \neq 0$. With it,

$$v^T B u = -k_1 \frac{T - m_a g}{m_a} + k_2 \frac{L}{J_x} + k_3 \frac{M}{J_y} + k_4 \frac{N}{J_z}. \tag{19}$$

By *Lemma 3*, if G is not an interior point of Ω, then $u = 0$ is not an interior point of $\mathcal{U}$. Then, there is a nonzero $k_u = [k_{u1}\ \ k_{u2}\ \ k_{u3}\ \ k_{u4}]^T$ satisfying

$$k_u^T u = k_{u1}(T - m_a g) + k_{u2} L + k_{u3} M + k_{u4} N \leq 0$$

for all $u \in \mathcal{U}$. Let

$$k = [-k_{u1} m_a\ \ k_{u2} J_x\ \ k_{u3} J_y\ \ k_{u4} J_z]^T \tag{20}$$

then $v^T B u \leq 0$ for all $u \in \mathcal{U}$ according to (19), which contradicts *Theorem 1*.

(ii)⇒(i): As all the eigenvectors of A^T are expressed in the form expressed by equation (18), then

$$v^T B u = k^T J_f^{-1} u$$

according to equation (1) and (18) where $k \neq 0$. Then there is no nonzero $v \in \mathbb{R}^8$ expressed by (18) satisfying $v^T B u \leq 0$ for all $u \in \mathcal{U}$ is equivalent to that there is no nonzero $k \in \mathbb{R}^4$ satisfying $k^T J_f^{-1} u \leq 0$ for all $u \in \mathcal{U}$. Supposing that (ii) is valid, then $u = 0$ is an interior point of $\mathcal{U}$. There is a neighbourhood $\mathcal{B}(0, u_r)$ of $u = 0$ belonging to $\mathcal{U}$, where $u_r > 0$ is small and constant. (ii)⇒(i) will be proved by counterexamples.

Supposing that condition (i) does not hold, then there is a $k \neq 0$ satisfying $k^T J_f^{-1} u \leq 0$ for all $u \in \mathcal{U}$. Without loss of generality, let $k = [k_1 \ * \ * \ *]^T$ where $k_1 \neq 0$ and $*$ indicates an arbitrary real number. Let $u_1 = [\varepsilon \ 0 \ 0 \ 0]^T$ and $u_2 = [-\varepsilon \ 0 \ 0 \ 0]^T$ where $\varepsilon > 0$; then $u_1, u_2 \in \mathcal{B}(0, u_r)$ if ε is sufficiently small. As $k^T J_f^{-1} u \leq 0$ for all $u \in \mathcal{B}(0, u_r)$, then $k^T J_f^{-1} u_1 \leq 0$ and $k^T J_f^{-1} u_2 \leq 0$. According to equation (1),

$$-\frac{k_1 \varepsilon}{m_a} \leq 0, \frac{k_1 \varepsilon}{m_a} \leq 0.$$

This implies that $k_1 = 0$ which contradicts the fact that $k_1 \neq 0$.

Then, condition (i) holds.

(ii)⇔(iii): According to the definition of $\rho(G, \partial\Omega)$, if $\rho(G, \partial\Omega) \leq 0$, then G is not in the interior of Ω, and if $\rho(G, \partial\Omega) > 0$, then G is an interior point of Ω.

This completes the proof.

B. *Proof of Theorem 3*

Theorem 3 will be proved in the following 3 steps.

Step 1. Obtain the equations (25), which are the projection of parallel boundaries in $\mathcal{F}$ by the map B_f.

The results in [17] are referred to in order to complete this step. First, (3) is rearranged as follows:

$$F = \begin{bmatrix} B_{1,j} & B_{2,j} \end{bmatrix} \begin{bmatrix} f_{1,j} \\ f_{2,j} \end{bmatrix} \tag{21}$$

where $f_{1,j} = [f_{S_1(j,1)} \ f_{S_1(j,2)} \ f_{S_1(j,3)}]^T \in \mathbb{R}^3$, $f_{2,j} = [f_{S_2(j,1)} \ \cdots \ f_{S_2(j,m-3)}]^T \in \mathbb{R}^{m-3}$, $j = 1, \cdots, s_m$. Write (21) more simply as

$$F = B_{1,j} f_{1,j} + B_{2,j} f_{2,j} \tag{22}$$

If the rank of $B_{1,j}$ is 3, there exists a 4 dimensional vector ξ_j such that

$$\xi_j^T B_{1,j} = 0, \|\xi_j\| = 1.$$

Therefore, multiplying ξ_j^T on both sides of (22) results in

$$\xi_j^T F - \xi_j^T B_{2,j} f_{2,j} = 0. \tag{23}$$

According to [17], $\partial\Omega$ is a set of hyperplane segments, and each hyperplane segment in $\partial\Omega$ is the projection of a 3 dimensional boundary hyperplane segment of $\mathcal{F}$. Each 3 dimensional boundary of the hypercube $\mathcal{F}$ can be characterized by fixing the values of $f_{2,j}$ at the boundary value, denoted by $\bar{f}_{2,j}$, where

$$\bar{f}_{2,j} \in \Pi_{i=1}^{m-3} \left\{0, K_{S_2(j,i)}\right\} \tag{24}$$

and allowing the values of $f_{1,j}$ to vary between their limits given by $\mathcal{F}$, where $f_{1,j} \in \Pi_{i=1}^{3} \left[0, K_{S_1(j,i)}\right]$. Then for each j, if rank $B_{1,j} = 3$, a group of parallel hyperplane segments $\Gamma_{\Omega,j} = \left\{l_{\Omega,j,k}, k = 1, \cdots, 2^{m-3}\right\}$ in Ω is obtained, and each $l_{\Omega,j,k}$ is expressed by

$$l_{\Omega,j,k} = \left\{X | \xi_j^T X - \xi_j^T B_{2,j} \bar{f}_{2,j} = 0, X \in \Omega, \bar{f}_{2,j} \in \Pi_{i=1}^{m-3} \left\{0, K_{S_2(j,i)}\right\}\right\} \tag{25}$$

where ξ_j is the normal vector of the hyperplane segments.

Step 2. Compute the distances from the center F_c to all the elements of $\partial\Omega$.

It is pointed out that, not all the hyperplane segments in $\Gamma_{\Omega,j}$ specified by equations (25) belong to $\partial\Omega$. In fact, for each j, only two hyperplane segments specified by equations (25) belong to $\partial\Omega$, denoted by $\Gamma_{\Omega,j,1}$ and $\Gamma_{\Omega,j,2}$, $j \in \{1, \cdots, s_m\}$, which are symmetric about the center F_c of Ω. The center of $\mathcal{F}$ is f_c, then F_c is the projection of f_c through the map B_f and is expressed as follows

$$F_c = B_f f_c \tag{26}$$

where $f_c = \frac{1}{2}[K_1\ K_2\ \cdots\ K_m]^T \in \mathbb{R}^m$. Then the distances from F_c to the hyperplane segments given by (25) are computed by

$$\begin{aligned} d_{\Omega,j,k} &= \left|\xi_j^T F_c - \xi_j^T B_{2,j} \bar{f}_{2,j}\right| \\ &= \left|\xi_j^T B_{2,j} \left(\bar{f}_{2,j} - f_{c,2}\right)\right| \\ &= \left|\xi_j^T B_{2,j} \bar{z}_j\right| \end{aligned} \tag{27}$$

where $k = 1, \cdots, 2^{m-3}$, $f_{c,2} = \frac{1}{2}[K_{S_2(j,1)}\ K_{S_2(j,2)}\ \cdots\ K_{S_2(j,m-3)}]^T \in \mathbb{R}^{m-3}$, $\bar{f}_{2,j}$ is specified by (24), and $\bar{z}_j = \bar{f}_{2,j} - f_{c,2}$.

Remark 4. The distances from F_c to the hyperplane segments given by (25) are defined by $d_{\Omega,j,k} = \min\left\{\|X - F_c\|, X \in l_{\Omega,j,k}\right\}$, $k = 1, \cdots, 2^{m-3}$.

The distances from the center F_c to $\Gamma_{\Omega,j,1}$ and $\Gamma_{\Omega,j,2}$ are equal, which is given by

$$d_{j,\max} = \max\left\{d_{\Omega,j,k}, k = 1, \cdots, 2^{m-3}\right\} \tag{28}$$

Since $\bar{z}_j \in Z = \frac{1}{2}\Pi_{i=1}^{m-3}\left\{-K_{S_2(j,i)}, K_{S_2(j,i)}\right\}, k = 1, \cdots, 2^{m-3}$,

$$d_{j,\max} = \frac{1}{2}\text{sign}\left(\xi_j^T B_{2,j}\right) \Lambda_j \left(\xi_j^T B_{2,j}\right)^T \tag{29}$$

according to (12) (27) and (28), where Λ_j is given by (15).

Step 3. Compute $\rho\left(G, \partial\Omega\right)$.

As G and F_c are known, the vector $F_{Gc} = F_c - G$ is projected along the direction ξ_j and the projection is given by

$$d_{Gc} = \xi_j^T F_{Gc}. \tag{30}$$

Then if $G \in \Omega$, the minimum of the distances from G to both $\Gamma_{\Omega,j,1}$ and $\Gamma_{\Omega,j,2}$ is

$$d_j = d_{j,\max} - |d_{Gc}| \tag{31}$$

But if $G \in \Omega^C$, d_j specified by (31) may be negative. So the minimum of the distances from G to both $\Gamma_{\Omega,j,1}$ and $\Gamma_{\Omega,j,2}$ is $|d_j|$. According to (26) (29) (30) and (31),

$$d_j = \frac{1}{2}\text{sign}\left(\xi_j^T B_{2,j}\right)\Lambda_j\left(\xi_j^T B_{2,j}\right)^T - \left|\xi_j^T\left(B_f f_c - G\right)\right|, j = 1, \cdots, s_m.$$

But if rank $B_{1,j} < 3$, the 3 dimensional hyperplane segments are planes, lines, or points in $\partial\Omega$ or Ω and $|d_j|$ will never be the minimum in $|d_1|$, $|d_2|$, $\cdots$, $|d_{s_m}|$. The distance d_j is set to $+\infty$ if rank $B_{1,j} < 3$. The purpose of this is to exclude d_j from $|d_1|$, $|d_2|$, $\cdots$, $|d_{s_m}|$. In practice, $+\infty$ is replaced by a sufficiently large positive number (for example, $d_j = 10^6$). If $\min\left(d_1, d_2, \cdots, d_{s_m}\right) \geq 0$, then $G \in \Omega$ and $\rho\left(G, \partial\Omega\right) = \min\left(d_1, d_2, \cdots, d_{s_m}\right)$. But if $\min\left(d_1, d_2, \cdots, d_{s_m}\right) < 0$, which implies that at least one of $d_j < 0, j \in \{1, \cdots, s_m\}$, then $G \in \Omega^C$ and $\rho\left(G, \partial\Omega\right) = -\min\left(|d_1|, |d_2|, \cdots, |d_{s_m}|\right)$ according to (9).

Then $\rho\left(G, \partial\Omega\right)$ is computed by

$$\rho\left(G, \partial\Omega\right) = \text{sign}\left(\min\left(d_1, d_2, \cdots, d_{s_m}\right)\right)\min\left(|d_1|, |d_2|, \cdots, |d_{s_m}|\right). \tag{32}$$

This is consistent with the definition in (9).

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