

Factorisation of Rational Motions: A Survey with Examples and Applications

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Abstract

Since its introduction in 2012, the factorization theory for rational motions quickly evolved and found applications in theoretical and applied mechanism science. We provide an accessible introduction to motion factorization with many examples, summarize recent developments and hint at some new applications. In particular, we provide pseudo-code for the generic factorization algorithm, demonstrate how to find a replacement linkage for a special case in the synthesis of Bennett mechanisms and, as an example of non-generic factorization, synthesize open chains for circular and elliptic translations.

Keywords: motion polynomial, dual quaternions, mechanism synthesis, Bennett flip, multiplication trick

1 Introduction

The factorization of rational motions was introduced in [1, 2] and has ever since proved its usefulness in mechanism synthesis. So far, it has been used for the construction of new linkages [2–8] with attractive properties for applications, including rational parameterisations of all relative motions in the linkage, computable (sometimes even linear) input-output relations, and the availability of simple symbolic and numeric synthesis algorithms.

The fundamental factorization theorem of [2] is a statement about “generic” rational motions. Current research focuses on non-generic cases. Preliminary results are quite promising and will also be addressed in this survey article. Related results can be found in [8] (these proceedings).

We continue this paper with a brief introduction to dual quaternions and motion polynomials in Section 2. It is followed by algorithms and examples obtained from factorization in generic cases (Section 3). In particular, we discuss a special case in the synthesis of Bennett mechanisms where the two 2R dyads coincide and no valid 4R linkage can be constructed. It is, however, possible to find a 5R or 6R linkage that generates the same motion.

In Section 4 we discuss circular and elliptic translations as an exceptional case of motion factorization. Here, the generic factorization algorithm fails but straightforward computations give a result. By means of a recently invented “multiplication trick” we can generate non-planar 4R chains for elliptic translations and construct linkages from them.

2 Dual Quaternions and Motion Polynomials

Quaternions and dual quaternions already proved their usefulness in computational kinematics and mechanism science. See [9], [10, Ch. 4] or [11, Ch. 9] for an introduction. Here, we briefly summarize some basic definitions and concepts that will be needed in later sections.

2.1 Quaternions and dual quaternions

A quaternion can be written as $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ with real numbers q_0, q_1, q_2, q_3 . The set of all quaternions is denoted by \mathbb{H} . Quaternions are added component-wise and their multiplication is determined by the rules

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

It is associative but not commutative. We have, for example $(\mathbf{i} + \mathbf{j})(2 - \mathbf{k}) = 2\mathbf{i} + 2\mathbf{j} - \mathbf{ik} - \mathbf{jk} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{j} - \mathbf{i} = \mathbf{i} + 3\mathbf{j}$ but $(2 - \mathbf{k})(\mathbf{i} + \mathbf{j}) = 2\mathbf{i} + 2\mathbf{j} - \mathbf{ki} - \mathbf{kj} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{j} + \mathbf{i} = 3\mathbf{i} + \mathbf{j}$.

The conjugate quaternion to $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ is $\bar{q} = q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}$. Depending on the individual author, the quaternion norm is either $q\bar{q} = q_0^2 + q_1^2 + q_2^2 + q_3^2 \in \mathbb{R}$ or the square root of $q\bar{q}$. We follow the first convention. Quaternions with unit norm $q\bar{q} = 1$ can model rotations in $\text{SO}(3)$ via

$$x = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \mapsto y_1\mathbf{i} + y_2\mathbf{j} + y_3\mathbf{k} = qx\bar{q}. \quad (1)$$

Here, a vector $(x_1, x_2, x_3) \in \mathbb{R}^3$ is identified with the vector quaternion $x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \in \mathbb{H}$. The oriented axis direction

of the displacement (1) is (q_1, q_2, q_3) and the rotation angle φ satisfies $\tan(\varphi/2) = q_0(q_1^2 + q_2^2 + q_3^2)^{-1/2}$. Note that q and $-q$ describe the same rotation.

A dual quaternion can be written as $h = p + \varepsilon q$ with $p, q \in \mathbb{H}$. The *dual unit* ε satisfies $\varepsilon^2 = 0$ and commutes with $\mathbf{i}, \mathbf{j}, \mathbf{k}$. For example, $(\mathbf{i} + \varepsilon \mathbf{j})(\mathbf{j} - \varepsilon \mathbf{k}) = \mathbf{ij} + \varepsilon(\mathbf{j}^2 - \mathbf{ik}) + \varepsilon^2 \mathbf{jk} = \mathbf{k} - \varepsilon(1 - \mathbf{j})$. The set of dual quaternions is denoted by \mathbb{DH} , the conjugate dual quaternion is $\bar{h} = \bar{p} + \varepsilon \bar{q}$, and the dual quaternion norm is $h\bar{h} = p\bar{p} + \varepsilon(p\bar{q} + q\bar{p})$. Because of $p\bar{p}, p\bar{q} + q\bar{p} \in \mathbb{R}$, it is a dual number. An important difference between quaternions and dual quaternions is that $h \neq 0$ does not imply existence of $h^{-1} \in \mathbb{DH}$ such that $hh^{-1} = 1$. Dual quaternions different from the set $\{\varepsilon d \mid d \in \mathbb{H}\}$ do not possess such a multiplicative inverse. If h is not from that set, its multiplicative inverse is $h^{-1} = (h\bar{h})^{-1}\bar{h}$. It equals the conjugate up to a scalar dual factor.

Unit dual quaternions model SE(3), the group of rigid body displacements. Identifying the point (x_1, x_2, x_3) with the dual quaternion $x = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$, the action of the unit dual quaternion $h = p + \varepsilon q$ is

$$x \mapsto px\bar{p} + p\bar{q} - q\bar{p} = px\bar{p} + 2p\bar{q}. \quad (2)$$

We see that the spherical component of the rigid body displacement is described by the primal part p while the translation is given by the vector quaternion $p\bar{q} - q\bar{p} = 2p\bar{q}$.

Again, h and $-h$ model the same rigid body displacement. This suggests, to projectivise the space of dual quaternions by identifying h and λh for any $\lambda \in \mathbb{R} \setminus \{0\}$. This turns \mathbb{DH} into the real projective space P^7 of dimension seven. All points (classes of proportional dual quaternions) that can be normalized to unity describe rigid body displacements. These are precisely the points of the quadric hypersurface $S \subset P^7$ given by $p\bar{q} + q\bar{p} = 0$ (the *Study quadric*) minus the three-space given by $p = 0$ (the *exceptional three space E*).

2.2 Polynomials and Motion Polynomials

The set of polynomials in t with coefficients in \mathbb{DH} is denoted by $\mathbb{DH}[t]$. Polynomial multiplication is defined by the convention that the indeterminate t commutes with all coefficients. For $h, k \in \mathbb{DH}$ we have, for example

$$C = (t - h)(t - k) = t^2 - (h + k)t + hk. \quad (3)$$

The value of the polynomial $C = \sum_{i=0}^n c_i t^i \in \mathbb{DH}[t]$ at $h \in \mathbb{DH}$ is defined as $C(h) := \sum_{i=0}^n c_i h^i$. Note that with this convention we have for the polynomial C in (3)

$$\begin{aligned} C(h) &= h^2 - (h + k)h + hk = hk - kh, \\ C(k) &= k^2 - (h + k)k + hk = -hk + hk = 0. \end{aligned}$$

This seemingly strange behavior is a consequence of the definition for multiplication and evaluation. Linear factors of polynomials over \mathbb{DH} are not directly related to zeros. There is, however, an important exception (Lemma 2 of [2]):

Proposition 1 *The polynomial $C \in \mathbb{DH}[t]$ can be factored as $C = C'(t - h)$ with $C' \in \mathbb{DH}[t]$ if and only if $C(h) = 0$.*

In other words, the zeros of C correspond to linear *right factors* of C .

The conjugate of the polynomial $C = P + \varepsilon Q = \sum_{i=0}^n c_i t^i$ is defined as $\bar{C} = \sum_{i=0}^n \bar{c}_i t^i$ and the norm polynomial is $C\bar{C} = P\bar{P} + \varepsilon(P\bar{Q} + Q\bar{P})$. Its coefficients are dual numbers. A polynomial $C = P + \varepsilon Q \in \mathbb{DH}[t]$ is called a *motion polynomial*, if its leading coefficient is invertible and the norm polynomial is even real. This important condition ensures that the curve parameterised by $C(t)$ for $t \in \mathbb{R}$ lies in the Study quadric. Hence motion polynomials act via (2) on points $x = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ and yield rational parametric equations

$$t \mapsto \frac{Px\bar{P} + P\bar{Q} - Q\bar{P}}{P\bar{P}} = \frac{Px\bar{P} + 2P\bar{Q}}{P\bar{P}}, \quad t \in \mathbb{R} \cup \{\infty\}$$

for all trajectories. The extension of the parameter range to $\mathbb{R} \cup \{\infty\}$ via $C(\infty) = c_n$ is necessary for obtaining complete curves. Summarizing, we can say that *motion polynomials parameterise rational motions*. Conversely, it follows from [12] that any motion with only rational trajectories is given by a suitable motion polynomial.

Lets look at linear motion polynomials $C = c_1 t + c_0$. Because c_1 is invertible, we may transform it to $C = t - h$ by left multiplying with c_1^{-1} . This amounts to a change in the fixed coordinate frame and is no loss of generality. The norm polynomial is $(t - h)(t - \bar{h}) = t^2 - (h + \bar{h})t + h\bar{h}$ and the conditions for a linear motion polynomial are

$$h + \bar{h} \in \mathbb{R}, \quad h\bar{h} \in \mathbb{R}.$$

The second condition implies that h itself describes a rigid body displacement, the first condition implies vanishing of the dual scalar part. In coordinates we have $h = h_0 + h_1\mathbf{i} + h_2\mathbf{j} + h_3\mathbf{k} + \varepsilon(h_5\mathbf{i} + h_6\mathbf{j} + h_7\mathbf{k})$ with $h_1h_5 + h_2h_6 + h_3h_7 = 0$. It is known that the rigid body displacements $C(t), t \in \mathbb{R} \cup \{\infty\}$ are either all rotations about the fixed axis with Plücker coordinates $[h_1, h_2, h_3, -h_5, -h_6, -h_7]$ or, if $h_1 = h_2 = h_3 = 0$, translations in direction (h_5, h_6, h_7) .

Remark 2 In case of a rotation, the rotation angle φ satisfies $\tan(\varphi/2) = (t - h_0)(h_1^2 + h_2^2 + h_3^2)^{-1/2}$. Using the coefficients $a = -h - \bar{h}$ and $b = h\bar{h}$ of the norm polynomial, this becomes $\tan(\varphi/2) = (2t + a)(4b - a^2)^{-1/2}$. *The rotation angle at the parameter value t is hence already determined by the norm polynomial of $t - h$.* A similar statement holds true for the translation distance in case of $h_1 = h_2 = h_3 = 0$. Also note that h is a zero of the norm polynomial $(t - h)(t - \bar{h})$.

3 Generic Factorization

A motion polynomial $C = P + \varepsilon D$ is called *generic*, if its primal part P has no real factors. If P has a real factor with

(possibly complex) zero t_0 , the point $C(t_0)$ lies in the exceptional generator E of the *Study quadric* S . Hence, C is not generic, if it intersects E (possibly over the complex numbers) or, equivalently, if the degree of the spherical motion component, which is parameterised by P , can be reduced by dividing off a real factor. The fundamental theorem in motion factorization is

Proposition 3 ([2], Theorem 1) *If C is a monic, generic motion polynomial of degree n , there exist rotation polynomials $t - h_1, \dots, t - h_n$ such that*

$$C = (t - h_1) \cdots (t - h_n). \quad (4)$$

We call (4) a factorization of C . If C is not generic, factorizations of the shape (4) may or may not exist. It is also noteworthy and very important for applications in mechanism science that the factorization (4) is, in general, not unique (compare Algorithm 2 below). It can be viewed as a decomposition of a rational motion into the product of rotations, whose angles are linked via the common parameter t . In other words, every factorization of the motion polynomial C corresponds to an open nR chain whose end-effector can follow the motion parameterised by C . Combining several of these open chains will in general result in an over-constrained linkage with end-effector motion parameterised by C .

3.1 Computation

Computing a factorization (4) of a generic motion polynomial $C = P + \varepsilon Q$ is not much harder than the factorization of real polynomials over the complex numbers. An important ingredient is right division for polynomials in $\mathbb{D}\mathbb{H}[t]$. Given A and $B \in \mathbb{D}\mathbb{H}[t]$ such that the leading coefficient of B is invertible, there exist unique polynomials Q and R , called *quotient* and *remainder*, such that $A = QB + R$ and $\deg R < \deg B$. Algorithm 1 computes Q and R under the additional assumption that B is monic. This is no real loss of generality and is usually fulfilled in our applications. Note that there is no essential difference between Algorithm 1 and a standard algorithm for polynomial division in $\mathbb{R}[t]$. In particular, it does not require any quaternion division. Because of non-commutativity it is, however, important to strictly adhere to the given order of factors (for example, Bc in Line 6 must not be replaced by cB).

Algorithm 2 can be used to compute a factorization of a generic, monic motion polynomial C . (If the leading coefficient c_n of C is different from one, we can compute a factorization of $c_n^{-1}C$ and left-multiply it with c_n .) By genericity, the norm polynomial $C\bar{C}$ has no real zeros. Hence, there exist irreducible quadratic polynomials $M_1, \dots, M_n \in \mathbb{R}[t]$ such that $C\bar{C} = M_1 \cdots M_n$. The factorizations of C are labeled by the permutations of these factors. The functions $\text{REMAINDER}(C, M_n)$ in Line 3 and $\text{QUOTIENT}(C, t - h_i)$ in Line 6 are derived from Algorithm 1. In Line 4 we have to

Algorithm 1 Polynomial division

Require: $A, B \in \mathbb{D}\mathbb{H}[t]$, B monic.

Ensure: Q, R such that $A = QB + R$ and $\deg R < \deg B$.

```

1:  $Q \leftarrow 0, R \leftarrow A$ 
2: while  $\deg(R) \geq \deg(B)$  do
3:    $c \leftarrow$  leading coefficient of  $R$ 
4:    $n \leftarrow \deg(R) - \deg(B)$ 
5:    $Q \leftarrow Q + ct^n$ 
6:    $R \leftarrow R - Bct^n$ 
7: end while
8: return  $Q, R$ 

```

compute the zero $r_1^{-1}r_0$ of a linear polynomial $R = r_1t + r_0$. It exists and is unique if r_1 is invertible. Again, this is guaranteed by genericity of C .

From Algorithm 2 we infer further properties of motion factorization. They follow from the labeling of factorizations by quadratic factors of $C\bar{C}$ and Remark 2.

Proposition 4 *In general, a motion polynomial of degree n admits $n!$ different factorizations. In any factorization, the factors labeled by the same quadratic factor M_i of the norm polynomial give rise to rotations with the same angular velocity.*

Algorithm 2 Factorization of motion polynomials

Require: Generic, monic motion polynomial $C \in \mathbb{D}\mathbb{H}[t]$ of degree n , tuple $F = (M_1, \dots, M_n)$ of quadratic, irreducible real factors of $C\bar{C}$.

Ensure: List $H = [h_1, \dots, h_n]$ such that $C = (t - h_1) \cdots (t - h_n)$ and $M_i = (t - h_i)(t - \bar{h}_i)$ for $i = 1, \dots, n$; initially $H = []$ is the empty list.

```

1: repeat
2:    $F \leftarrow (M_1, \dots, M_{n-1})$ 
3:    $R \leftarrow \text{REMAINDER}(C, M_n)$ 
4:    $h \leftarrow$  unique zero of  $R$ 
5:    $H \leftarrow [h, H]$  ▷ prepend  $h$  to list
6:    $C \leftarrow \text{QUOTIENT}(C, t - h_i)$ 
7: until  $\deg C = 0$ 
8: return  $H$ .

```

Even if most examples of motion factorization in literature are symbolic, it is important to remark that both Algorithm 1 and Algorithm 2 can easily be implemented with floating point numbers. From a numerical point of view, the most challenging step in the computation is the factorization of the norm polynomial $C\bar{C} = M_1 \cdots M_n$. But this is a standard problem of numerical mathematics with a variety of fast and stable algorithms available.

3.2 Synthesis of Bennett linkages

The most important application of motion factorization is linkage synthesis. As one particular example, we present the synthesis of spatial four-bar linkages (Bennett linkages). We are not content with the generic case alone but also discuss special cases that might occur. Synthesis algorithms for Bennett linkages are known for a while, see [13] or [14] and the references therein. The approach via motion factorization gives additional insights and, more importantly, allows generalizations as in [5].

We wish to construct a Bennett linkage that visits three given poses p_0, p_1, p_2 which we view as points in $S \setminus E$. A crucial observation of [13] is that the coupler motion of a Bennett linkage is parameterised by a quadratic motion polynomial that can easily be computed. Lets consider the example

$$\begin{aligned} p_2 &= 1 - \mathbf{i} - \mathbf{j} - \mathbf{k} + \varepsilon(1 + \mathbf{k}), \\ p_1 &= 3 - \mathbf{i} - 2\mathbf{j} - \mathbf{k} + \varepsilon(1 - \mathbf{i} + \mathbf{j} + 2\mathbf{k}), \\ p_2 &= 1. \end{aligned}$$

Setting $C = t^2 + (\lambda p_1 - 1 - \mu p_0)t + \mu p_0$ with yet undetermined $\lambda, \mu \in \mathbb{R} \setminus \{0\}$ we ensure $C(0) = \mu p_0 \equiv p_0, C(1)\lambda p_1 \equiv p_1, C(\infty) \equiv p_2$. Here, the symbol “ \equiv ” denotes equality in projective sense, that is, up to scalar factors. Now we have to select λ and μ such that the dual part of the norm polynomial $C\bar{C}$ vanishes. Comparing coefficients, we obtain the equations

$$\lambda - \mu = 2\mu - \lambda\mu - \lambda = \mu(\lambda - 1) = 0$$

with the unique solution $\lambda = \mu = 1$. Thus, the coupler motion of the sought Bennett mechanism is given by the quadratic motion polynomial

$$C = t^2 + (1 - \mathbf{j})t + 1 - \mathbf{i} - \mathbf{j} - \mathbf{k} - \varepsilon((\mathbf{i} - \mathbf{j} - \mathbf{k})t - 1 - \mathbf{k}).$$

It is rather typical for this approach to determine the coupler motion before the dimension and position of the actual mechanism (“mechanism from motion”). This can be done by computing the two factorizations of C . In order to use Algorithm 2, we need the quadratic, irreducible factors M_1 and M_2 of $C\bar{C}$. In our example, we obtain

$$M_1 = t^2 + 2, \quad M_2 = t^2 + 2t + 2.$$

Running Algorithm 2 twice with $F = (M_1, M_2)$ and $F = (M_2, M_1)$ yields the two factorizations $C = (t - h_1)(t - h_2) = (t - k_1)(t - k_2)$ where

$$\begin{aligned} h_1 &= \mathbf{j} + \mathbf{k} + \varepsilon(\mathbf{i} + \mathbf{j} - \mathbf{k}), & h_2 &= -1 - \mathbf{k} - 2\varepsilon\mathbf{j}, \\ k_1 &= -1 - \mathbf{i} - \varepsilon\mathbf{k}, & k_2 &= \mathbf{i} + \mathbf{j} + \varepsilon(\mathbf{i} - \mathbf{j}). \end{aligned} \quad (5)$$

The quaternions h_2 and k_2 describe rotations about the moving axes in the reference configuration $C(\infty) = 1$ (the identity). The quaternions h_1 and k_1 give the position of the fixed

axes in that configuration. The linkage’s Denavit-Hartenberg parameters can easily be computed from the Plücker coordinates of the moving and fixed revolute axes. The loop closure equation is

$$\begin{aligned} 1 &= (t - h_1)(t - h_2)(t - k_2)^{-1}(t - k_1)^{-1} \\ &\equiv (t - h_1)(t - h_2)(t - \bar{k}_2)(t - \bar{k}_1) \\ &= (t^2 + 2)(t^2 + 2t + 2). \end{aligned}$$

where “ \equiv ” denotes equality up to real polynomial factors.

We conclude this section with a hint to [5]. This paper presents a synthesis method for over-constrained 6R linkages to four prescribed poses. It is based on rational cubic interpolation on the *Study quadric* S and subsequent motion factorization and can be viewed as direct generalization of the synthesis of Bennett linkages we just presented. The interpolation procedure is, however, more involved: Interpolants exist only if the three-space spanned by the given poses intersects the Study quadric S in a ruled quadric (a hyperboloid, not an ellipsoid). Then, two one-parametric families of rational interpolating cubic curves exist, each giving rise, via motion factorization, to families of solution linkages.

3.3 What Can Go Wrong?

Above example presented an ideal case for synthesis of Bennett linkages. But things can go wrong in special situations. Usually, literature on synthesis of Bennett linkages ignores these exceptional cases but a robust synthesis procedure should be aware of them.

To begin with, the curve parameterized by C may actually lie on a straight line. In this case, the end-effector motion is either a rotation about a fixed axis or a straight line. Situations like this are easy to detect and we exclude them in the following. Should C parameterise a planar or spherical motion, the factorization algorithm will work and yield a planar or spherical anti-parallelogram linkage. Note, however, that in these cases a two-parametric set of quadratic interpolants C exists, because the dual part of $C\bar{C}$ vanishes identically in λ and μ .

The conditions for using Algorithm 2 may not be met, that is, the primal part P of $C = P + \varepsilon Q$ may have a real factor. If P is even real, the motion is a curvilinear translation along a rational curve of degree two. We do not intend to give a full discussion here. In Section 4 we will deal in detail with the cases of elliptic and circular translations. If P has precisely one real factor of degree one, there exists a parameter value t_0 with $P(t_0) = 0$ and hence $C(t_0) = \varepsilon Q(t_0)$. Should $Q(t_0)$ be zero, the curve lies on a straight line. Hence $Q(t_0) \neq 0$ and $C(t_0)$ lies in the exceptional generator of the *Study quadric* S . This case has been discussed in [15]. It leads to the coupler motion of RPRP linkages [16] and is still amenable to Algorithm 2 — even if not all prerequisites are met. An

example is $C = (t - h_1)(t - h_2) = (t - k_1)(t - k_2)$ where

$$\begin{aligned} h_1 &= \mathbf{i} + \mathbf{j} + \varepsilon(\mathbf{i} - \mathbf{j}), & h_2 &= 1 + \varepsilon(\mathbf{i} + 3\mathbf{j} - \mathbf{k}), \\ k_1 &= 1 + \varepsilon(3\mathbf{i} + \mathbf{j} - \mathbf{k}), & k_2 &= \mathbf{i} + \mathbf{j} - \varepsilon(\mathbf{i} - \mathbf{j}). \end{aligned}$$

Note that h_1 and k_2 are rotation quaternions with (necessarily) parallel axes while h_2 and k_1 are translation quaternions.

Finally, we also would like to discuss a situation that was pointed out by M. Hamann (Dresden University of Applied Sciences). It is possible that Algorithm 2 is applicable but does not yield a proper linkage because the two synthesized 2R dyads *coincide*. Using motion factorization, we can easily characterize this situation and construct a replacement mechanism.

It is known [17] that the coupler motion of Bennett linkages is obtained by reflecting a fixed system in the rulings of one family on a hyperboloid of revolution (it is *line symmetric* with respect to that system of rulings). Using the hyperboloid with equation $x^2b^2c^2 + y^2a^2c^2 - z^2a^2b^2 = a^2b^2c^2$ and $a, b, c \in \mathbb{R} \setminus \{0\}$, a possible motion polynomial reads

$$\begin{aligned} C &= (b^2 + c^2)t^2 \\ &+ (2a(c\mathbf{j} + b\mathbf{k}) - 2\varepsilon(a^2(b\mathbf{j} - c\mathbf{k}) - bc(b\mathbf{k} + c\mathbf{j})))t \quad (6) \\ &+ 2bc\mathbf{i} - b^2 + c^2 - 2a\varepsilon((b^2 - c^2)\mathbf{i} + 2bc) \end{aligned}$$

Two factorizations $C = (b^2 + c^2)(t - h_1)(t - h_2) = (b^2 + c^2)(t - k_1)(t - k_2)$ can only coincide if the norm polynomial factors as $\overline{CC} = M^2$ with a quadratic, irreducible polynomial $M \in \mathbb{R}[t]$. From the discriminant

$$4096(b^2 + c^2)^8(a^2 + c^2)^2(a + b)^2(a - b)^2 \quad (7)$$

of \overline{CC} we read off that this can only happen if $a = b$. Indeed, we then have $\overline{CC} = (a^2 + c^2)^2(t^2 + 1)^2$. Thus, we have

Proposition 5 *A line symmetric motion with respect to one family of lines on a hyperboloid H can be generated by only one 2R dyad if and only if H is a hyperboloid of revolution. Otherwise, there exist two 2R dyads.*

In other words, synthesis of a Bennett linkage fails if the three input poses are sampled from a line symmetric motion with respect to a hyperboloid of revolution.

3.4 Bennett flips

Motion factorization not only reveals a caveat of existing Bennett synthesis methods but can also be used for the construction of a replacement linkage with five or six joints. This construction is based on an important technique for rational linkages which we call *Bennett flip*. Consider the motion polynomial $C = (t - h_1)(t - h_2)$, pick an arbitrary rotation quaternion p and use Algorithm 2 to compute rotation quaternions f, g, r, q such that

$$\begin{aligned} (t - h_1)(t - p) &= (t - f)(t - g), \\ (t - \overline{h_2})(t - p) &= (t - q)(t - \overline{r}). \end{aligned}$$

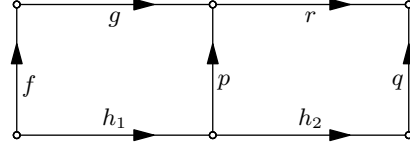


Figure 1: Link graph for construction of replacement mechanisms

The link diagram of this construction is depicted in Figure 1. The vertices indicate links and the arrows with label x connect links such that the relative motion of the second (at the arrow tip) with respect to the first (at the arrow base) is parameterised by $t - x$. We have the loop closure equations

$$\begin{aligned} 1 &\equiv (t - h_1)(t - p)(t - \overline{g})(t - \overline{f}) \\ &\equiv (t - q)(t - \overline{r})(t - \overline{p})(t - h_2). \end{aligned} \quad (8)$$

The second closure equation implies $(t - q)(t - \overline{r}) \equiv (t - \overline{h_2})(t - p)$ so that

$$\begin{aligned} &(t - h_1)(t - h_2)(t - q)(t - \overline{r})(t - \overline{g})(t - \overline{f}) \\ &\equiv (t - h_1)(t - h_2)(t - \overline{h_2})(t - p)(t - \overline{g})(t - \overline{f}) \quad (9) \\ &\equiv (t - h_1)(t - p)(t - \overline{g})(t - \overline{f}) \\ &\equiv 1. \end{aligned}$$

The last equality follows from the first closure equation in (8). From (9) we infer that the rotation quaternions $h_1, h_2, q, \overline{r}, \overline{g}, \overline{f}$ can be assembled, in that order, to form an over-constrained 6R linkage. As long as p is chosen sufficiently general, this construction works and the resulting 6R linkage (of type Waldron's double Bennett hybrid) has precisely one degree of freedom. The link connecting h_2 and q performs the motion parameterized by C . Note that this construction does not hinge on the vanishing of the discriminant (7) but is valid for arbitrary Bennett motions.

A special choice of p may even cause the axis of g and r to coincide. This then yields an over-constrained 5R linkage (a Goldberg mechanism), for which $(t - h_1)(t - h_2)$ parameterises the motion of one link. Let's look at the concrete example obtained by plugging $a = b = 1$ and $c = 2$ into the motion polynomial (6). It has the factorization $C = (t - h_1)(t - h_2)$ where

$$h_1 = -\frac{1}{5}(4\mathbf{j} - 3\mathbf{k} + 2\varepsilon(3\mathbf{j} + 4\mathbf{k})), \quad h_2 = -\mathbf{k}.$$

Setting $p = \mathbf{j} - \mathbf{k} + \frac{5}{2}\varepsilon(\mathbf{j} + \mathbf{k})$ we compute the rotation quaternions f, g, r , and q via Bennett flips:

$$\begin{aligned} f &= \frac{1}{5}(\mathbf{j} - 7\mathbf{k}) + \frac{9}{10}\varepsilon(7\mathbf{j} + \mathbf{k}), \\ g &= \mathbf{k} - 5\varepsilon\mathbf{j} = r, \\ q &= \mathbf{j} + \mathbf{k} - \frac{5}{2}\varepsilon(\mathbf{j} - \mathbf{k}). \end{aligned}$$

Because $g = r$, two consecutive axes coincide and the corresponding linkage has only five revolute joints.

4 Exceptional Factorizations

Motion polynomials whose primal part has no real factor can always be factorized by Algorithm 2. However, many rational motions of practical and theoretical importance are not generic. For those motions, computation of the unique zero h_i of a linear remainder polynomial in Line 4 of Algorithm 2 may fail. Note that motion polynomials in $\mathbb{H}[t]$ are always projectively equal to generic motion polynomials and Algorithm 2 always works. This is an example for the failure of naive transfer of results of spherical kinematics to spatial kinematics via wrong application of Kotelnikov's and Study's principle of transfer.

Failure of Algorithm 2 does not imply anything on existence or non-existence of factorizations. There exist rational motions with no factorization and rational motions with infinitely many factorization.

4.1 Elliptic translations

Let us discuss, as an example, the motion polynomial

$$C = t^2 + 1 + \varepsilon(b\mathbf{j}t + a\mathbf{i}), \quad a \geq b > 0. \quad (10)$$

It parameterizes an elliptic translation. The primal part of C is real so that Algorithm 2 is not applicable. But we can try to manually solve the equation

$$C = (t - h_1)(t - h_2) \quad (11)$$

for $h_1 = p_1 + \varepsilon d_1$, $h_2 = p_2 + \varepsilon d_2$. Comparing coefficients of the primal part, we immediately find the conditions $p_1 + p_2 = 0$, $p_1 p_2 = 1$ and the solution

$$p_1 = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}, \quad p_2 = \overline{p_1} \quad \text{where} \quad x^2 + y^2 + z^2 = 1.$$

The dual parts are $d_i = u_i\mathbf{i} + v_i\mathbf{j} + w_i\mathbf{k}$ for $i = 1, 2$. Comparing coefficients gives the additional equations

$$\begin{aligned} (u_1 - u_2)x + (v_1 - v_2)y + (w_1 - w_2)z &= 0 \\ (v_1 + v_2)z - (w_1 + w_2)y + a &= 0 \\ (u_1 + u_2)z - (w_1 + w_2)x &= 0 \\ (u_1 + u_2)y - (v_1 + v_2)x &= 0 \\ u_1 + u_2 = b + v_1 + v_2 = w_1 + w_2 &= 0. \end{aligned} \quad (12)$$

The system (12) has the solution

$$\begin{aligned} x = 0, \quad z = \frac{a}{b}, \quad u_1 = \lambda, \quad u_2 = -\lambda, \quad v_1 = \mu - b, \quad v_2 = -\mu, \\ w_1 = \frac{yb(b - 2\mu)}{2a}, \quad w_2 = -\frac{yb(b - 2\mu)}{2a}, \end{aligned}$$

with $\lambda, \mu \in \mathbb{R}$. From $x^2 + y^2 + z^2 = 1$ and $a \geq b > 0$ we infer $a = b$ and $y = 0$ whence the solution becomes

$$\begin{aligned} x = y = 0, \quad z = 1, \quad u_1 = \lambda, \quad u_2 = -\lambda, \\ v_1 = \mu - a, \quad v_2 = -\mu, \quad w_1 = 0, \quad w_2 = 0. \end{aligned}$$

In summary, we have shown:

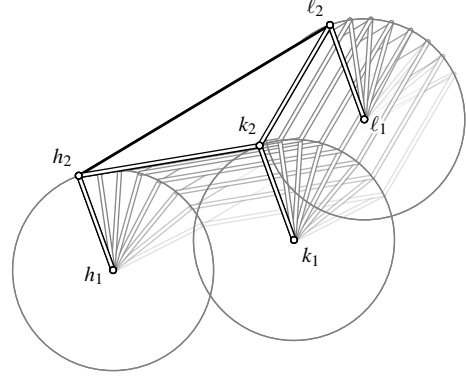


Figure 2: Three different factorizations of a circular translation

Proposition 6 *Elliptic translations do not admit factorizations of the shape $(t - h_1)(t - h_2)$ but circular translations admit infinitely many. If the circular translation is parameterized as in (10) with $a = b$, these factorizations are given by $h_1 = \mathbf{k} + \varepsilon(\lambda\mathbf{i} + (\mu - a)\mathbf{j})$, $h_2 = -\mathbf{k} - \varepsilon(\lambda\mathbf{i} + \mu\mathbf{j})$.*

In fact, it is well-known that a circular translation occurs in infinitely many ways as coupler motion of a parallelogram linkage. Each factorization corresponds to the motion of one 2R chain in such a linkage. This is illustrated in Figure 2.

4.2 The Multiplication Trick

The elliptic translation (10) admits no factorization with two linear motion polynomials. But there is a way to obtain factorizations with four motion polynomials. The basic idea is to find a real, irreducible polynomial Q such that QC admits a factorization. Note that C and QC are equal in projective sense, that is, they parameterise the same motion. By a result of [18], this is always possible and the degree of the real polynomial Q is bounded by the degree of the greatest real factor in the primal part of C .

Returning to the elliptic translation (10) and look at one particular example. Assuming $a \neq b$, we let $Q = t^2 + 1$ and compute $QC = (t - h_1)(t - h_2)(t - h_3)(t - h_4)$ where

$$\begin{aligned} h_1 &= \mathbf{k} - \frac{a-b}{a+b}\varepsilon\mathbf{i}, \\ h_2 &= -\mathbf{k} + \frac{(2(a-b)\mathbf{i} - (a+b)^2\mathbf{j})\varepsilon}{2(a+b)}, \\ h_3 &= -\mathbf{k} - (\mathbf{i} - \frac{1}{2}(a-b)\mathbf{j})\varepsilon, \\ h_4 &= \mathbf{k} + \mathbf{i}\varepsilon. \end{aligned} \quad (13)$$

By a construction similar to the Bennett flip in Section 3.4, we may “brace” each link h_i with an anti-parallelogram linkage with joints h_i, m_i, k_i, m_{i-1} . Combining these anti-parallelograms gives a linkage where the link connecting m_4 and h_4 performs an elliptic translation (Figure 3). The construction is initialized by picking a generic planar rotation

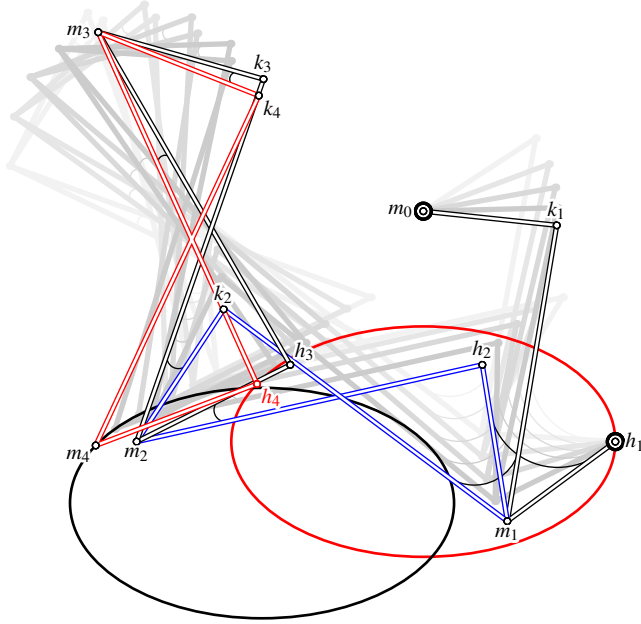


Figure 3: Linkage to generate an elliptic translation and corresponding link graph

quaternion m_0 and recursively computing k_i and m_i from

$$(t - m_{i-1}^{-1})(t - h_i) = (t - k_i)(t - m_i^{-1}), \quad i = 1, 2, 3, 4.$$

The joints triplets h_i, m_i, h_{i+1} and k_i, m_i, k_{i+1} , respectively, belong to the same link. This is indicated by circular arcs in Figure 3. Constructions of this type are at the core of [19] where bounded rational curves of arbitrary degree are drawn by planar linkages.

The factorization (13) is by no means unique. We also have the strange factorization $QC = (t - k_1)(t - k_2)(t - k_3)(t - k_4)$ where

$$\begin{aligned} k_1 &= -\frac{(a^2 - b^2)\mathbf{i} - 2ab\mathbf{k}}{a^2 + b^2} - \frac{(a^2 - b^2)\boldsymbol{\varepsilon}\mathbf{j}}{a^2 + b^2}, \\ k_2 &= \frac{(a^2 - b^2)\mathbf{i} - 2ab\mathbf{k}}{a^2 + b^2} - \frac{((a^2 + b^2)^2 - 2b(a^2 - b^2))\boldsymbol{\varepsilon}\mathbf{j}}{2b(a^2 + b^2)}, \\ k_3 &= -\mathbf{i} + \frac{a^2 - b^2 + 2b}{2b}\boldsymbol{\varepsilon}\mathbf{j}, \\ k_4 &= \mathbf{i} - \boldsymbol{\varepsilon}\mathbf{j}. \end{aligned}$$

In contrast to (13), this factorization is *not planar*. It gives rise to a spatial open chain with parallel first and second and parallel third and fourth axis that can follow a curvilinear translation along an elliptic path. We may combine the four

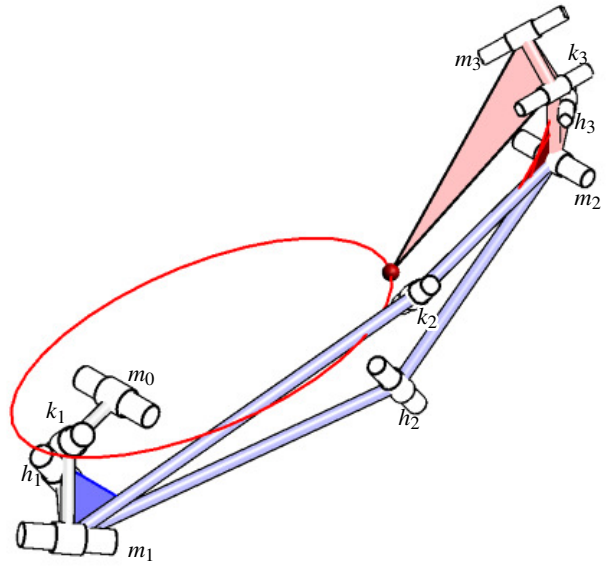


Figure 4: Spatial linkage to generate a Darboux motion.

revolute axes of the latter factorization with two prismatic joints to obtain an over-constrained RRRRPP mechanism one of whose links performs an elliptic translation.

If we abandon the requirement to generate an elliptic translation, we may also make the motion polynomial C of (10) factorizable by right-multiplying it with a suitable quaternion polynomial $H \in \mathbb{H}[t]$. This changes the motion but not the trajectory of the origin in the moving coordinate frame. If H is linear, the motion polynomial CH parameterizes a Darboux motion (a spatial motion with only planar trajectories) and our Bennett flip construction produces a spatial linkage to draw an ellipse (Figure 4). The anti-parallelograms of Figure 3 are replaced by Bennett linkages and only ten joints and eight links are required. While the linkage has theoretical full-cycle mobility, it is, of course, a demanding engineering task to realize this in a practical design. For more information about Darboux linkages we refer to [8]. There, also considerable refinements of the above construction can be found.

5 Conclusions

This article gives an overview of algorithms and techniques associated with the factorization of motion polynomials. It addresses several special but yet instructive cases and gives concrete applications. We complemented known methods for the synthesis of Bennett linkages by a discussion of a special case that so far has skipped attention of researchers and we showed how to use Bennett flips to produce replacement linkages. We also hinted at a new method for the construction of rational over-constrained 6R linkages to four prescribed poses. The attentive reader may have noticed a

gap in the chain of synthesis procedures: We did not talk about 5R linkages. In fact, it is possible to synthesis 5R linkages to three poses plus “some additional constraints”. An efficient algorithm to do this hinges on a better understanding of three-spaces generated by 2R dyads and will be addressed in future work.

In a second part, we gave a preview on very recent methods to factor exceptional motion polynomials that evade the algorithms for generic situations. The “multiplication trick” is actually a powerful tool to factor all motion polynomials and, together with Bennett flips, belongs to a toolbox for the construction of linkages to arbitrary rational motions or to arbitrary rational trajectories. The general theory and algorithmic treatment of the “multiplication trick” is currently being worked out. As these theoretical constructions do not necessarily require an excessive number of links, there is hope for useful engineering applications.

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